



## Core Knowledge, Language, and Number

Elizabeth S. Spelke

To cite this article: Elizabeth S. Spelke (2017) Core Knowledge, Language, and Number, Language Learning and Development, 13:2, 147-170, DOI: [10.1080/15475441.2016.1263572](https://doi.org/10.1080/15475441.2016.1263572)

To link to this article: <http://dx.doi.org/10.1080/15475441.2016.1263572>



Published online: 16 Mar 2017.



Submit your article to this journal [↗](#)



Article views: 115



View related articles [↗](#)



View Crossmark data [↗](#)

ARTICLES

## Core Knowledge, Language, and Number

Elizabeth S. Spelke

Department of Psychology, Harvard University, Cambridge, Massachusetts, USA

### ABSTRACT

The natural numbers may be our simplest, most useful, and best-studied abstract concepts, but their origins are debated. I consider this debate in the context of the proposal, by Gallistel and Gelman, that natural number system is a product of cognitive evolution and the proposal, by Carey, that it is a product of human cultural history. I offer a third proposal that builds on aspects of these views but rejects one tenet that they share: the thesis that counting is central to number. I suggest that children discover the natural numbers when they learn a natural language: especially nouns, number words, and the rules that compose quantified noun phrases. This learning, in turn, depends both on cognitive systems that are innate and shared by other animals, and on our species-specific language faculty. Thus, natural number concepts are unique to humans and culturally universal, yet they are learned.

Natural number concepts may be our simplest abstract ideas. These concepts are exceedingly useful, serving as a basis for measurement, money, and mathematics, orienting us in space and time, and structuring activities from sports to elections. Likely because of their simplicity and ubiquity, the development of natural number concepts has been richly studied since the landmark research of Jean Piaget (1952) and the enduring challenges to his theory that followed (e.g., Gelman, 1972; Mehler & Bever, 1967; Siegal, 1999). Despite the simplicity of these concepts and the large body of research probing their development, however, the psychological foundations of natural number concepts continue to be debated.

In the context of this debate, it is useful to characterize the natural number system in three interconnected ways. First, there is a minimal unit, ONE, that corresponds to the smallest distance separating distinct numbers (hereafter, the UNIT principle). Second, natural numbers can be generated by successive addition of one (hereafter, the principle of SUCCESSION<sup>1</sup>). Third, two sets whose members can be placed in one-to-one correspondence have the same cardinal value: they are equal in number (hereafter, the principle of EXACT EQUALITY). Here I consider three general accounts of the development of this system of concepts, in relation to research that focuses on key aspects of these principles.

According to the first account, natural number concepts are part of human nature. They are ancient: they evolved in distant ancestors, and so we share them with other animals. They are innate and begin functioning early in development: their emergence is not shaped by encounters with sets

**CONTACT** Elizabeth S. Spelke  [spelke@wjh.harvard.edu](mailto:spelke@wjh.harvard.edu)  Department of Psychology, Harvard University, 33 Kirkland St., Cambridge, MA 02138.

<sup>1</sup>In the literature on numerical development in children, the acquisition of “the successor principle” refers to a milestone in children’s mastery of counting: the point at which children understand that every word in their verbal count list refers to a cardinal value that is one larger than the value designated by the previous word. In some discussions, mastery of the successor principle of verbal counting is considered as a criterion for mastery of the natural number system (e.g., Sarnecka, 2016). Use of this criterion, however, would beg the present question. Because numerical language is learned, natural number concepts that were defined by children’s learning of number word meanings necessarily would be learned, ruling out Gelman and Gallistel’s nativist claims by definition and rendering Carey’s claims true by definition. On pain of circularity, experiments testing these theories and others require characterizations of numerical concepts that are independent of children’s learning of language.

and their transformations but instead serves to structure those encounters. And, these concepts are present in all human cultures: later development builds on them but does not overturn them, so they are available to children and adults everywhere. According to the second account, natural number concepts depend on a specific product of culture: a counting procedure. These concepts are recent and unique to humans, because the first counting procedure appears to have been invented relatively late in human prehistory. They are learned: indeed, contemporary children master counting procedures slowly and with difficulty. And they are culturally variable: different human groups count in different ways and to different extents, and some groups do not count at all.

Drawing on old and new research, I suggest that neither of these theories captures the development of natural number concepts, and I sketch a third account of their emergence. I propose that natural number concepts arise through the productive combination of representations from a set of innate, ancient, and developmentally invariant cognitive systems: systems of core knowledge. In particular, natural number concepts depend on a system for representing sets and their approximate numerical magnitudes (hereafter, the Approximate Number System (ANS)) and a set of systems that collectively serve to represent objects as members of kinds. None of these core systems is unique to humans, but their productive combination depends on the acquisition and use of a natural language. Because both the core systems and the language faculty are universal across humans, and because children master their native language spontaneously, natural number concepts emerge universally, with no formal or informal instruction. Because language is unique to humans, so is our grasp of the natural numbers. Finally, because specific natural languages are learned, the system of natural number concepts is neither innate nor present in the youngest children.

I begin by describing the first core system, the ANS. Then I turn to Gelman and Gallistel's theory, considering both its virtues and the problems it faces in accounting for some prominent limits to the numerical reasoning of young children. These limits suggest that children's earliest numerical representations fail to capture two of the three key principles that characterize the natural numbers: principles of exact equality and succession. That suggestion, in turn, motivates Carey's account of the emergence of natural number concepts. After considering some strengths of this account, I turn to findings that raise problems for it. Then I suggest how language learning, rather than either the innate unfolding of a genetic program or the acquisition of culture-specific counting devices, might allow children to discover the natural numbers.

## A core system of number

From newborn infants to professional mathematicians, humans spontaneously represent the cardinal values of sets of objects or sequences of events with ratio-limited precision (see Dehaene, 2011, for review). Experiments conducted on newborn infants serve to illustrate this ability (Izard, Sann, Spelke, & Streri, 2009). Infants in a maternity hospital were familiarized with sequences of syllables. The particular syllables changed from one sequence to the next, as did the duration of the syllable sequences, but the number of syllables was constant—4 long syllables per sequence for half the infants and 12 shorter syllables per sequence for the others. While infants listened to the sequences, alternating visual arrays appeared, containing 4 or 12 objects of variable but comparable shapes and sizes. Infants looked reliably longer at the object array that corresponded in number to the sequence of sounds. Because the visual and auditory arrays differed in modality and format, and could not be matched on the basis of continuous, extensive, or intensive quantities such as contour length, sequence duration, or item size, these looking patterns provide evidence that infants detected the numerical correspondence between these arrays, and therefore the numerical distinction between 4 and 12. Further experiments revealed that newborn infants also distinguish 6 from 18 but not 4 from 8. Beginning in the first days after birth, the ability to discriminate between two numbers depends on their ratio.

Do newborn infants relate sequences of syllables to arrays of visual forms through noisy processes of one-to-one correspondence, applied to the individual members of these sets? Alternatively, do

infants represent the sequences and arrays as ensembles with approximate numerical magnitudes, matching sets with similar magnitudes? Several findings indicate that the latter representations underlie infants' performance. Processing of individual objects shows a set size limit—most adults cannot hold more than four objects in mind at once, and infants are limited to fewer objects than this (Oakes, Baumgartner, Barrett, Messenger, & Luck, 2013)—but infants respond to number in much larger arrays. Indeed, infants often fail to represent the numerical sizes of very small sets of objects. For example, newborn infants fail to match sequences of 2 sounds to arrays of 2 rather than 6 objects when tested under the same conditions that reveal matching of sequences of 4 sounds to arrays of 4 rather than 12 objects (Coubart, Izard, Spelke, Marie, & Streri, 2014). When presented with very small sets, infants and adults alike typically focus attention on the individual objects and suppress representations of the set's numerical magnitude (Hyde & Spelke, 2009, 2011), although both adults and infants can focus on the numerical values of small sets if presentation conditions make individual objects difficult to track (Hyde & Wood, 2011; Starr, Libertus, & Brannon, 2013). Newborn infants' numerical representations therefore do not depend on, and may not even allow, selective attention to the individual members of the sets to which they apply.

Most studies of sensitivity to number have been conducted with older infants, who detect number in diverse displays including action sequences (Wood & Spelke, 2005), clouds of dots (e.g., Xu & Spelke, 2000), and arrays of 3D objects (Xu & Garcia, 2008). Infants look longer at numerically changing than numerically constant displays (Libertus & Brannon, 2010), and they distinguish increases from decreases in number (Brannon, 2002; de Hevia & Spelke, 2010). Infants also add and subtract successively presented arrays to arrive at rough estimates of their sum or difference (McCrink & Wynn, 2004). Infants match arrays that increase in number to objects that increase in length, even as newborns (de Hevia, Izard, Coubart, Spelke, & Streri, 2014). Finally, infants are sensitive to the proportionate sizes of subsets of objects, in arrays containing large numbers of objects of two or more colors (McCrink & Wynn, 2007). They may use these proportions to estimate the probability that individual objects of each color will be randomly sampled (Denison & Xu, 2010).

The findings from studies of infants accord with a rich literature on human adults, who show the same patterns of performance, with the same signature ratio limit, when presented with numerical displays under conditions that preclude counting or subitizing (e.g., Barth, Kanwisher, & Spelke, 2003; Barth et al., 2006; Hyde & Spelke, 2009; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004). Moreover, common brain regions respond to number in infants and adults (for reviews see Dehaene, 2011; and Dehaene-Lambertz & Spelke, 2015). In light of these findings, it is perhaps not surprising that evidence for ANS representations has been obtained from adults living in diverse cultures, with or without any formal education (Gordon, 2004; Piazza, Pica, Izard, Spelke, & Dehaene, 2013; Pica, Lemer, Izard, & Dehaene, 2004).

The findings from studies of infants also accord with a large body of experiments investigating numerical cognition in non-human animals. In particular, monkeys discriminate between arrays of visible objects or sequences of sounds on the basis of number, with ratio-limited accuracy (e.g., Cantlon & Brannon, 2007). Their accuracy improves with training (e.g., Brannon & Terrace, 1998), and training effects generalize across arrays differing in modality and format (e.g., Nieder, Freedman, & Miller, 2002). Monkeys can be trained to perform approximate addition and subtraction (Cantlon & Brannon, 2007), and they spontaneously match arrays that vary in number with lines that vary in length (Tudusciuc & Nieder, 2009). Finally, monkeys and humans show common neural signatures of numerical representations (Nieder & Dehaene, 2009).

How does this core system of number relate to the symbolic number systems that children learn in school, and that adults use to perform formal mathematics? Is the ANS an evolutionary curiosity, with little application to the modern world in which number is used precisely, or does it support the development and use of natural number concepts? Although there is persisting controversy concerning the role of the ANS in mathematical reasoning, a wealth of research links ANS representations to representations of the natural numbers. Here I focus on three findings.

First, young children who have learned the meanings of the first three number words, but not counting, show a brain response indicative of the detection of incongruity when a number word accompanies a visual display presenting the wrong number of objects, and the magnitude of the response scales with the size of the numerical mismatch, in accord with the ratio signature of the ANS (Pinhas, Donohue, Woldorff, & Brannon, 2014). This finding suggests that children learn the meanings of the first number words, in part, by mapping those words to representations of approximate number. Second, after children learn to identify words for numbers beyond 20 but before they begin to study arithmetic, they are able to solve verbal problems of approximate symbolic addition and subtraction (e.g., “This girl has 15 marbles and gets 10 more; this boy has 42 marbles, who has more marbles?”), and their performance shows signatures of the ANS (Gilmore, McCarthy, & Spelke, 2007). Third, the brain regions that respond to approximate number in tasks that activate the ANS overlap significantly with the brain regions that are activated during performance of symbolic mathematical tasks, in adults (e.g., Piazza, Pinel, Le Bihan, & Dehaene, 2007), children (Cantlon et al., 2009), and even mathematicians (Amalric & Dehaene, 2016).

Common patterns of performance and of brain activity do not reveal, however, whether the ANS plays a causal role in the development or use of symbolic number concepts. The best evidence addressing this question comes from studies using training methods. Park and Brannon (2013) trained separate groups of adults on problems of non-symbolic numerical ordering, non-symbolic numerical addition, (using arrays of dots in each case), symbolic numerical ordering, or a non-numerical task. Adults trained for 10 days on non-symbolic addition showed enhanced performance on a test of symbolic arithmetic, compared to each of the other groups, but not on any symbolic tests outside the domain of mathematics, consistent with the possibility that exercise of approximate, non-symbolic arithmetic enhances symbolic arithmetic performance.

In parallel, Hyde, Khanum, and Spelke (2014) trained separate groups of 6- to 7-year-old children on 4 different non-symbolic tasks: numerical comparison, numerical addition of 2 successively visible arrays that children were asked to imagine as forming one larger set, comparison of brightness values in a single object, and addition of line lengths in 2 successively visible, elongated lines that children were asked to imagine as forming one longer line. Children trained on either of the two numerical tasks showed better performance of symbolic arithmetic, relative to children trained on either non-numerical task, but no enhancement on a reading test of comparable difficulty. These effects were replicated in a separate study, conducted with children in Pakistan (Khanum, Hanif, Spelke, Berteletti, & Hyde, 2016). Children who are learning symbolic arithmetic therefore benefit, in some way, from ANS practice.

What is the source of this benefit? It is possible that ANS training produces a placebo effect (Boot, Simons, Stothart, & Stutts, 2013): children may expect that tasks training attention to number (but not to brightness or length) will enhance their arithmetic performance (but not their reading), and this expectation may boost their performance.<sup>2</sup> Thus, we presented Hyde et al.’s (2014) tests to a new group of children, drawn from the same population (Dillon, Pires, Hyde, & Spelke, 2015). After taking each outcome test, children practiced each of the four training tasks briefly, and then they were asked whether they thought that performance on each outcome measure would benefit from practice on that task, as well as from a good night’s sleep or a good meal. Children reliably judged that they would perform better on the symbolic arithmetic test if they were well fed rather than hungry and well rested rather than tired, but they did not expect to perform better after practicing the two tasks that exercised the ANS, relative to the other tasks. These findings suggest that the ANS training did not induce a placebo effect. Instead, exercise of the ANS appears specifically to enhance children’s performance of mathematics. It is possible, however, that this effect is transitory: no training study yet reveals enduring benefits of ANS training on children’s learning of mathematics.

---

<sup>2</sup>Placebo effects are less plausible in Park & Brannon’s experiments, because their participants showed benefits on only one of three numerical tasks.

In summary, the core system of number found in human infants appears to have many of the features of the system of natural number: It is abstract over variations in modality and format, and it supports numerical operations of comparison, addition, subtraction, and mapping of numbers to lines. Finally, the ANS connects in some way to children's learning of number symbols and their performance of symbolic arithmetic. All of these findings cohere with the first theory of number development, to which I turn.

### Is the natural number system a product of core knowledge?

Gelman and Gallistel (1978) launched the modern study of children's numerical development with landmark studies of the conceptual foundations of children's counting. Gelman found that young children, given a set of objects, count them flexibly and systematically. Asked to start their count with a different object, they comply, cognizant of the fact that the same items can be counted in different orders. In contrast, children reject counts that violate the invariant order of the words in their count list or that repeat or skip words or objects. Remarkably, some children adhered to a stably ordered count list before they learned the full conventional list, suggesting that their ordering is not a product of rote learning or instruction (Gelman & Gallistel, 1978). Gelman and Gallistel (1978) proposed that children are endowed with a core set of principles that generate the natural numbers. In later work, the authors connected those principles to the ANS, either operating alone (Gallistel, 1990) or together with capacities to perform logical operations on an innate concept ONE (Leslie, Gelman, & Gallistel, 2008).<sup>3</sup>

Nativist theories of the natural numbers account well for the ubiquity of the natural number system across human cultures (Butterworth, 1999). They also provide a natural explanation for children's widely observed ability to gain access to this system prior to the onset of formal education, simply by observing or interacting with people who use it. Such theories are countered by reports of a single remote culture that appears to lack number words or a counting procedure (Gordon, 2004; Frank, Everett, Fedorenko, & Gibson, 2008; see also Everett, 2005), but those reports have been disputed (Butterworth, Reeve, Reynolds, & Lloyd, 2008; Nevins, Pesetsky, & Rodrigues, 2009).

Nativist theories also have been challenged by a large literature showing that children learn number word meanings slowly (e.g., Fuson, 1988; Wynn, 1990, 1992b). Much of this research has used a simple method: the "Give-N" task. First children are asked to count a set of objects, and the words they use are recorded (typically, the first ten number words). Then the children are shown a pile of objects (e.g., toy fish) and are asked for a specific number of them, using the number words that the children produced while counting (e.g., "Can you put two fish in the pond?"). Most of the children in these studies produce one fish on demand as early as two years of age, but they take many months to learn enough of the meaning of *two* to produce two fish on demand. Once they succeed with *two*, they will spend more months learning the meanings of *three* and sometimes *four* and even *five*, before they succeed with all the words in their count list. Thus, children engage in counting, and recite number words, long before they appear to understand that counting is a procedure of enumeration, or that number words refer to precise cardinal values.

These findings do not refute the claim that natural number concepts are innate, however, because children's slow, step-by-step learning of number words could reflect their difficulty mapping their number concepts onto language, rather than limitations to the number concepts themselves. When a young child hears *seven* applied to a set of seven objects, for example, the word will not be paired consistently with the same internally represented number, because of children's inability to enumerate sets of seven objects exactly. Children may be predisposed to map *seven* onto a set with an exact

<sup>3</sup>There are important differences between Gallistel and Gelman's earlier and later theories, as their thinking evolved within the growing field of numerical cognition, as well as differences between their theories and those of others who posit an innate system of natural number (Brannon & Merritt, 2011; Butterworth, 1999; Wynn, 1998). For present purposes, I ignore these differences and focus on common features of all the theories that posit innate access to the natural numbers.

cardinal value, but they likely will estimate the cardinal value of groups labeled *seven* sometimes as SIX or EIGHT, or even FIVE or NINE. Thus, children who possessed the full system of natural number concepts, comprising exact equality, succession, and the concept ONE, likely would not be able to determine the correct cardinal value given by each number word in their language until they learned to enumerate sets of objects exactly, by counting.

Counting, moreover, may be hard for children to learn, despite the ease with which children come to mimic it. Verbal counting is pragmatically opaque. To count a set of objects, one says each number word while pointing to a different object in the set, but the spoken words do not designate either the individual objects to which one points or the set as a whole. Instead, each word pronounced during counting gives the cardinal value of the subset of objects that one has pointed to thus far in the count sequence. Nothing in the counting routine makes this reference apparent. The verbal counting routine therefore presents children with a mapping problem that would be difficult to solve even if the children had a full understanding of the sequence of natural numbers (Fuson, 1988; Wynn, 1992b).

In brief, the evidence described thus far does not indicate whether infants and young children grasp the logic of natural number. The innateness of the natural number system can be tested only by devising situations in which young children confront problems whose solution requires the system of natural number concepts, but does not require an understanding of number words or counting procedures. Building on research by Jennifer Lipton (Lipton & Spelke, 2005), Veronique Izard created a test that goes a long way toward meeting these requirements (Izard, Streri, & Spelke, 2014). She asked whether children near their third birthday understand that two sets that can be placed in one-to-one correspondence have the same cardinal value (the principle of exact equality) and that the cardinal value of a set increases when one element is added to the set (as implied by the principle of succession).

In her studies, children's reasoning about exact numbers was probed through the use of a device that established a one-to-one correspondence between a set of objects and a set of anchoring locations. Children just under three years of age were introduced to a collection of finger puppets, each of which was placed on a different branch of a tree. In most conditions, children saw a tree with six branches, accompanied by either five or six featurally indistinguishable puppets. First the experimenter and child placed all the puppets on the tree, noting that each puppet occupied one branch (and, for the set of five, that one branch was empty). Then the experimenter and child together put all the puppets into a single opaque box and rocked them to sleep. Finally, children were asked to wake up the puppets and return them to the tree. On the trials on which the initial number of puppets was six, the experimenter surreptitiously removed one puppet from the box, so that children's search would always yield just five puppets. After the fifth puppet was retrieved, Izard recorded the child's time spent searching in the empty box. If children expected exactly five puppets on the tree, then they should have stopped searching after retrieving the fifth puppet. If they expected six or more puppets, then they should have searched further.

In this first condition, children distinguished reliably and appropriately between the sets of five and six puppets: in the latter case, they searched more persistently after placing the fifth puppet on the tree. Further conditions showed that children succeeded by assessing the one-to-one correspondence between puppets and branches: when the number of branches was increased to 11 (such that a tree now contained 5 or 6 empty branches as well as 5 or 6 branches with puppets), children failed to distinguish the two sets. Thus, children used one-to-one correspondence rules to reproduce sets of five or six objects.

In light of children's success at reproducing the original sets with six-branch trees, Izard repeated the experiments but introduced transformations to the puppets in the box. In one condition, six puppets entered the box and then one of the puppets left, or five puppets entered the box and then a sixth puppet joined them. The latter transformation, in particular, tested a key aspect of children's sensitivity to succession: If they inferred that adding one object to a set of objects resulted in a larger set, then they should have searched in the box longer after retrieving the fifth puppet when one

puppet was added to a set of five than either when no objects were added to the set of five, or when one object was removed from a set of six. A second condition was the same as the first but with smaller numbers: three puppets entered the box and then one left, or two puppets entered and one was added. Children's searching of the box and placement of the puppets on the tree were measured, as before, to assess how many puppets they expected to retrieve on each trial.

Children expected the correct number of puppets when the box contained two or three puppets and was transformed by addition or subtraction of one, showing that they understood the scenarios and could solve these simple addition and subtraction problems, as in past research (Brooks, Audet, & Barner, 2013; Condry & Spelke, 2008; Wynn, 1992a). In contrast, children failed to expect the correct number after the same addition or subtraction of one object from a box containing five or six objects. This failure contrasted with children's correct performance when five or six objects were shaken inside the box but none were added or removed. Children evidently can track individual objects over time, remember what happened to them, and determine the exact number of objects to return to the tree by relations of one-to-one correspondence with its branches. Nevertheless, children failed to infer that a set of five objects increased in cardinal value after the addition of one object.

Children's successful performance in the experiment with no addition or subtraction could be interpreted as indicating that children infer that one-to-one correspondence implies numerical equality, but the finding is open to an alternative interpretation: children may search the box appropriately for five vs. six puppets *not* because they expect a given number, but because they expect a given set of objects. Izard addressed this possibility in a final experiment in which children again viewed five or six puppets that were arrayed on a tree and then placed in a box. In this experiment, children witnessed two transformations of the box's contents: first one object was subtracted from the box, then one object was added. In one condition (identity), the same object participated in the subtraction and addition events: a single puppet left the box and then returned. In a second condition (substitution), one puppet left the box and a different (but featurally indistinguishable) puppet replaced it. The conditions were animated and worded so that the events were similar in complexity and duration, but the two puppets in the substitution condition came from different locations and were described as different individuals.

If children command the logic of exact numerical equality, they should infer that adding one object increments the value of a set by one, and that subtracting one object decrements the value of a set by one (see especially Leslie et al., 2008). Two outcomes, therefore, would be consistent with this theory. First, if children can keep track of the events, then they should expect the same number of objects at the end of the events as at the beginning, in both conditions. Second, if children cannot keep track both of the one-to-one correspondence of puppets to tree branches and of the two transformations, then they should fail in both conditions. Children should not perform differently in the identity and substitution conditions, if they have a concept of exact numerical equality, because natural number concepts apply to sets based on the number of elements that a set contains, regardless of the identities of the individuals that comprise it.

Contrary to the latter prediction, children's performance differed across these two conditions. When one object was removed from the box and then returned to it, children successfully restored the original quantity. In contrast, when one object was removed from the box and a different, featurally indistinguishable object replaced it, children searched equally for a sixth puppet whether the original number was six or five. Children evidently failed to appreciate that removing one object and replacing it with a different object restored the original cardinal value.

Together, these findings shed light both on children's numerical competence and on its limits. Young children reliably use one-to-one correspondence operations to construct a set composed of a specific, exact number of objects. Nevertheless, as children approach their third birthday, they fail to appreciate that the cardinal value of a set increases or decreases, respectively, after the addition or subtraction of one, contrary to the principle of succession. Moreover, children fail to appreciate that relations of one-to-one correspondence establish the numerical equivalence of two distinct sets of objects. Two sets of five objects are equivalent, for these children, if they contain the same

individuals, but not if they are composed of numerically distinct individuals, even if the individuals look the same and enter into the same one-to-one correspondence relations with other sets of individuals. Thus, children have a fairly well-developed notion of exact identity, but not a notion of exact numerical equality.

I believe these findings cast doubt on any theory of innate natural number concepts, because notions of exact numerical equality and succession stand at the heart of the system of natural number.<sup>4</sup> I turn, therefore, to Carey's theory of the construction of the natural numbers.

### Are natural number concepts a cultural construction?

Carey (2009) proposed that the system of natural number concepts is a joint product of humans' innate cognitive endowment (including our core knowledge systems and the capacities required for language learning) and our culture. There is nothing inevitable, for humans, about the natural numbers: there were times in the prehistory of our species when no one had these concepts, and there may be remote cultures that lack them today. Moreover, the natural numbers did not emerge suddenly in human cultural evolution but in a series of steps that included systems for distinguishing sets of one, two, and perhaps three, but not larger numbers; tally systems, in which each member of a set of pebbles or markings refers to an individual member of a set of countable entities; and finite lists of symbols, in which each symbol refers to a distinct, exact numerical magnitude, but the symbols do not combine to form larger magnitudes. All of these intermediate systems fall short of the infinite, productive symbol systems that are prevalent today, based primarily on counting procedures and on base systems that generate each larger number from smaller numbers through the operations of multiplication and addition.

On Carey's view, children's mastery of natural numbers mirrors in some ways the cultural history of number concepts. Contemporary children, living with educated adults, begin to break into the system of natural number in two parallel ways. First, they learn by rote to produce the symbols that figure in the counting routine. For most children, these are the ordered list of number words,<sup>5</sup> and they serve as placeholders: they have no content other than the content given by their ordering in the count list (*six* is the word after *five* and before *seven*), although they eventually gain further content from the syntax and semantics of the child's language (in English, *six* is a quantifier like *all* and *some*, and it appears in plural noun phrases). Second, children begin to master words that apply to sets composed of 1, 2, or 3 individuals. These words depend on explicit representations of the individuals: for example, *three* applies to an individual X, an individual Y distinct from X, and an individual Z distinct from X or Y, but not to an explicit numerical magnitude such as those delivered by the ANS. On Carey's view, representations from the ANS do not enter into children's representations of the meanings of number words until after children master the logic of counting.

Even adults have limited abilities to track multiple objects in parallel (e.g., Alvarez & Franconeri, 2007), and so the process by which children master the meanings of the first few number words cannot be extended indefinitely to account for their discovery of the meanings of words for large numbers. Carey argues that children gain access to larger number word meanings by making a deep analogy. *Two* is the word (or mark, or gesture) that comes after *one* and before *three* in children's ordered list of count words, and it designates an array of objects consisting of one more object than an array designated by *one* and one less object than an array designated by *three*. Children therefore reason that moving from any word to the next word in the count list is analogous to transforming one set into a different set by adding one new individual. Because the list of number words continues

<sup>4</sup>Although young children's failures in Izard's addition and substitution tasks provides evidence that they lack full natural number concepts, success would not imply that they command these concepts fully, because Izard's task does not test whether children have a concept ONE that separates every number from those that are closest to it (see Izard et al., 2008; and Rips et al., 2008; for discussion). I return to this question below.

<sup>5</sup>For simplicity, I assume in this exposition, that children are learning a verbal counting routine. Carey's theory applies, however, to any ordered list of symbols used in counting.

beyond *three*, children infer that the set of exact cardinal values continues as well, and that the Nth word in the count list refers to an array composed of N objects. The ANS plays no role in this induction, but later it is associated with the words used in counting, speeding children's numerical performance.

Carey's account encompasses a rich body of evidence, from human prehistory to contemporary anthropology, comparative linguistics, history of science, and cognitive development. A number of findings nevertheless pose problems for this account (see Spelke, [forthcoming](#), Ch. 9, for more discussion). First, recent studies suggest that children map number words to approximate numerical magnitudes well before they master the logic of counting, at about the same time that they master the meaning of *three* (Huang, Spelke, & Snedeker, 2010; Odic, Le Corre, & Halberda, 2015; Pinhas et al., 2014; Wagner & Johnson, 2011) or even earlier (Barner, 2016). Second, studies using the Give-N task with larger samples of children indicate that significant subsets of children learn individual number words up to five before they understand counting (Gunderson, Spaepen, & Levine, 2015; Posid & Cordes, 2015; Wagner & Johnson, 2011). It is unlikely that such children learn the meaning of *five* by representing five distinct individuals in parallel, as this ability lies beyond the powers of most adults, under most testing conditions.

The third finding pertains to the critical inductive leap that Carey posits. According to Carey, young children who learn the number words as placeholder concepts in the count list have access to the ordinal positions of the words in the memorized list, and they construct natural number concepts by mapping these ordinal relations (e.g., *three* comes after *two* in the list) to relations of relative number (*three* picks out a set with one more member than *two*). When adults and children learn verbal material by rote, however, neither the ordinal positions of words nor their relative order is explicitly available (Crowder & Greene, 2000; Hitch, Fastame, & Flude, 2005). The ordinal positions of number words remain inaccessible to young school-aged children well after they have mastered verbal counting. Elementary school children who use counting to add numbers tend to start their count with one: it takes considerable experience to discover that two can be added to five by starting with *five* and counting on (Siegler & Shrager, 1984). Even adults, tasked with judging which of two number words or symbols denotes a larger number, evidently rely on the associated numerical magnitudes delivered by the ANS: When previously numerate adults whose ANS is impaired by brain injury are asked whether *six* is bigger or smaller than *five*, they often respond by reciting the entire count list, beginning with "one." Although their ability to recite the count list survived their brain injury, their sense of the ordinal relations between items on the list did not.

These findings suggest that learning to recite the counting words in order does not make explicit the ordinal relations among these words. Consistent with this finding, studies by David Barner and his collaborators (Davidson, Eng, & Barner, 2012), show that even children who have learned to use counting to produce every number in their learned count list, on the Give-N task, often fail to judge that words occurring later in the count list refer to larger magnitudes, or that successive words in the count list differ by one. Learning to use counting to answer "how many?" questions evidently does not underlie mastery of these fundamental aspects of number word reference.

Studies of numerical development in Amazonian children raise further problems for Carey, as well as for Gelman and Gallistel (Jara-Ettinger, Piantadosi, Spelke, Levy, & Gibson, 2016). These studies used a variant of Izard's task with children living in the Bolivian Amazon: the Tsimane. The Tsimane language has an extremely limited numerical vocabulary and no counting routine, but Tsimane children learn Spanish as a second language in school, and they learn to count in Spanish. Because there is considerable variability in children's school attendance, this learning occurs at variable ages, as revealed by studies using the Give-N task (Piantadosi, Jara-Ettinger, & Gibson, 2014). Accordingly, Julian Jara-Ettinger gave the Give-N test to a group of Tsimane children, ranging in age from 4–11 years. By this test, some children knew no number words, some were one-, two-, three-, or four-knowers, and some had mastered the cardinal principle of counting.

He also tested the same children on a simplified version of Izard's task. First, he distributed 16 pictures of cookies in pairs to two cartoon characters, lining up the cookies in four rows of two;

consistent with the one-one correspondence operation and the pattern formed by the cookies, children judged that the two characters had the same number of cookies. Then he moved each set of eight cookies into a pile so as to remove the perceptual cues to numerical equality, and he performed one of six transformations on one of the two piles. On different trials, he removed half the cookies, rearranged the cookies without changing their number, removed one cookie, added one cookie, removed one cookie and then returned it to the pile (Izard's identity transformation), or removed one cookie from the pile and returned a different cookie (Izard's substitution transformation). Finally, he asked children whether the two characters still had the same number of cookies.

Children at all levels of number word knowledge tended to judge that the equality was disrupted when half the cookies were removed, and that it was maintained when the cookies were rearranged, showing that they understood the questions and were motivated to answer them correctly. When a single object was taken from or added to the pile, however, many children made errors, replicating Izard's finding that some children fail to expect the cardinal value of a set to increase with the addition of a single element. Also replicating Izard, errors were more frequent after the transformations of adding or removing one than after the identity transformation, and they were most frequent after the substitution transformation, even though this transformation, like the identity transformation, restored the original numerical value. For many Tsimane children, as for Izard's younger, U.S. children, the two numerically equivalent sequences of operations (subtracting one, then adding one) were understood differently, depending on the identity of the object(s) that were removed from and added to the pile. These Tsimane children failed to adhere to the principles of succession and exact equality.

Jara-Ettinger's primary analyses of the Tsimane children focused on the relation between children's number word knowledge and their performance on the +1, -1, identity, and substitution transformations. In general, errors declined as children's number word knowledge increased. However, there was no specific relation between mastery of counting (i.e., categorization of children as cardinal principle knowers on the Give-N test) and mastery of exact equality and succession (as indicated by perfect performance on the addition of 1, subtraction of 1, and substitution transformations). A significant subset of children who had mastered the logic of counting failed the tests probing their understanding of these fundamental aspects of the logic of natural number: mastery of counting did not suffice for mastery of this logic. Moreover, a significant subset of children who demonstrated mastery of the logic of exact equality and succession failed the test for mastery of counting: mastery of counting was not required for mastery of these basic principles. Thus, number word knowledge is predictive of successful understanding of natural number, but mastery of a counting procedure is neither necessary nor sufficient for the achievement of this understanding.

In summary, growing evidence suggests that counting is not the key to natural number. Learning to count, and to use counting to answer numerical questions, appears to be of limited help to children in discovering the natural numbers. Moreover, failing to learn counting appears to pose no barrier to this discovery.<sup>6</sup> These findings cast doubt on both of the theories discussed thus far, for both give a central place to counting. Might children gain access to the natural numbers in a different way?

## Do natural number concepts depend on language?

I propose that children's discovery of the natural numbers depends on their mastery of the generative rules of their language. As children learn a language, they discover the natural numbers in four steps. First, they

<sup>6</sup>One pillar of the natural number system—the concept of a minimal unit, ONE, that distinguishes each number from its nearest neighbors—is not tested by the tasks of Izard and Jara-Ettinger: although children who pass these tasks successfully infer that the cardinal value of a set is incremented by the addition of one element, and also that the operations of addition and subtraction of one element cancel one another, these studies do not reveal whether children infer that all numbers are separated from their nearest neighbors by the same minimal unit. This gap in the evidence does not bear on Carey's theory, because the central analogy that she posits presupposes the unit principle. After learning that *one*, *two*, and *three* designate sets that differ by exactly one element, Carey proposes that children infer that all the successive words in their count list will differ by exactly one element. Nothing inherent to the counting procedure licenses this inference, so Carey's theory requires that children apply the minimal unit principle prior to this learning (see Rips et al., 2008; and Izard et al., 2008; for discussion). I return to this principle in the next section (see especially footnote 13).

construct a prolific and productive system of representations of object kinds, by mastering noun phrases that, in English, are composed of determiners and singular count nouns that serve as sortals (e.g., *a cup, the cat, your hand*), naming the kind to which an object belongs (Spelke, *forthcoming*, Ch. 8).<sup>7</sup> Children master these noun phrases by linking each phrase to representations from three core systems: systems for representing cohesive, bounded objects, for representing animate agents and their actions, and for representing visual forms. Each noun phrase learned by young children refers to an entity with a characteristic function for itself (if it is an animal), its possessor (if it is a body part),<sup>8</sup> or its user (if it is an inanimate object). Most such entities also have a characteristic form that supports that function (see also Xu, 2007<sup>9</sup>).

Second, children learn natural language expressions that contain terms for individuals of different kinds (*the dog and the cat*) or individuals with different identities (*Evelyn and Simon; my cup and your cup; this dog, that dog, and the other one*). Using conjunction, these expressions refer to sets of two or three distinct individuals. They also support learning of new nouns at different taxonomic levels (e.g., *Look at those birds: there's a duck, a goose and a swan*). Finally, these expressions support learning of number word meanings (e.g., *John has three pets: a dog, a cat, and a hamster*). Thus, children can learn part of the meanings of the first number words as they occur in simple noun phrases referring to objects (*two dogs*), and eventually to entities of other kinds (*two oinks, three jumps*). Children discover that such noun phrases, like noun phrases with two or three conjuncts (*the big dog and the little one*), refer to two or three distinct individuals.

Third, children map these natural language expressions to representations from the ANS: they discover that expressions like *three dogs* and *Rover, Fido, and Lassie* refer to sets of a larger numerical magnitude than the sets picked out by expressions like *two dogs* or *Rover and Fido*. Such learning likely is hampered by the competitive relationship between ANS representations and representations of objects (e.g., Hyde & Wood, 2011), but language may reduce this competition in two ways. First, when children learn the meanings of expressions such as *two dogs*, this simple noun phrase can substitute for the active maintenance of parallel representations of two objects in working memory, thereby reducing or eliminating the inhibitory effects of object representations on the ANS. Second, learning of the mappings of the first three number words to ANS representations may benefit from expressions in which different number words are applied to sets of objects of the same kind, encouraging a comparison between different numerical magnitudes (e.g., *Do you want two cookies or three?*) These expressions may elicit ANS representations, because they present number words in contexts that highlight the numerical contrast.

Thus, children master the first three number words by mapping noun phrases containing the words both to representations of arrays of 1–3 individual objects and to representations of numerical magnitudes, delivered by the ANS. With these mappings, children can appreciate that *three dogs* refers to a set that is composed of exactly three distinct individuals and that is numerically larger than sets that can be referred to as *two dogs*.<sup>10</sup> Words for larger numbers cannot be represented in this manner, however. Limits to attention and working memory imply that children and adults can

<sup>7</sup>For ease of exposition, my discussion centers on NPs in English. Children who learn a language that lacks the count/mass and singular/plural distinction would construct kind representations by mastering other structures, such as classifiers.

<sup>8</sup>Body parts are not bounded objects, but names for body parts appear in children's earliest vocabulary and are invoked to represent space and number in many cultures. I suggest that a core system for representing agents also serves to represent body parts as entities with distinct forms and functions, as well as whole animate beings and their actions (Spelke, *forthcoming*, Ch. 3). These representations, together with representations of bounded objects and of geometrical forms, support children's early learning of nouns and noun phrases.

<sup>9</sup>Fei Xu's beautiful experiments provide much of the evidence for this claim, although she herself does not endorse it.

<sup>10</sup>Two questions arise here on which I take no stand. First, when do children begin to map noun phrases to ANS representations? Some evidence, discussed below, suggests that this mapping occurs only after children have mapped NPs containing *one, two, and three* to representations of 1–3 individuals of a kind. Recent evidence suggests, however, that this mapping may begin earlier, perhaps at the start of number word learning (Barner, 2016), when children first show evidence of representing sets of objects in non-verbal tasks (Feigenson, 2011; Feigenson & Halberda, 2008). The account I offer here is compatible with either of these possibilities. Second, where does the concept SET come from? If children begin to map NPs to ANS representations at the start of number word learning, then it may come from the ANS. Alternatively, it may come from the core object system (and therefore be available in representations of objects of specific kinds), from the language faculty, or from a domain-general core cognitive system supporting logical reasoning. In the absence of relevant evidence, I leave this question open. My account requires, however, that SET is available to children at the second step described above.

only reliably represent up to three or four individual objects in parallel. Moreover, limits to the precision of ANS representations imply that children and adults can only discriminate precisely the numerical magnitudes of small sets of objects.<sup>11</sup>

In the last step of the construction of natural number concepts, children apply the grammatical rules for forming expressions that refer to two or three individuals (*this dog and that one; two dogs*) so as to form expressions that refer to two or three sets of individuals (*three cows and one more, two geese and three ducks, two groups of three puppets*). Within a single noun phrase, children thereby construct representations of larger numbers from representations of smaller ones, through the mental operations of addition and multiplication. The sets to which these noun phrases refer inherit the properties of the sets that are the referents of noun phrases containing the first three number words: they are sets of individuals, with exact cardinal values that increase with the addition of one. Equipped with these expressions, children are in a position to learn new number words that designate the cardinal values of the sets picked out by these complex noun phrases (e.g., *three cows and two horses* designates a set composed of a specific number of animals: *five animals*). Each newly learned number word reduces the processing demands posed by expressions that combine words for smaller numbers, and each such word can enter into new expressions (e.g., *five puppets and one more; two groups of five puppets*), extending the range of numbers to which children's expressions can refer.

In summary, I suggest that the productive rules of natural languages, together with the resources of core knowledge, allow children to gain two critical insights. First, numerical expressions can be composed to express new numbers. These new expressions designate sets whose numerical magnitudes are the sum or product of the magnitudes of the sets that compose them: From *three horses* and *two dogs* comes *three animals and two more*; from *three horses* and *three cows* comes *two groups of three animals*. By forming and interpreting these expressions, children gain the insight that numbers can be added and multiplied to form other numbers. Just as *three* can be applied to dogs, barks, and jumps, it can be applied to numbers: *three pairs, three dozen, three hundred*. And just as *and one more* can be added to a singular expression, it can be added to composed numerical expressions recursively, in principle without limit. This insight does not depend either on the mastery of a counting procedure or on the deciphering of a conventional base system in languages that have one. As Chomsky (1988) proposed, the productive rules for forming an infinite set of discrete, natural language expressions may underlie the discovery of the infinite series of natural numbers.

By packaging objects of a kind into groups, mapping the numerical magnitude of each group to an ANS representation, and combining groups together, number words and expressions therefore convey to children that sets of one, two, or three individuals are not the only sets with exact cardinal values: every number word and expression can be applied to a set of entities with a precise cardinal value. Like *two dogs* and *three cats*, expressions like *two dogs and three cats* no longer apply to a set if an object is added to or removed from the set, even if the change in cardinal value is not immediately perceptible. By using productive language rules, children come to discover these relationships at the center of the system of natural number.

While children are working out the numerical relationships that their language makes explicit, language may foster the development of natural number concepts in two further ways. First, language expressions ease demands on children's working memory: although an array of five horses would equal or exceed the attentive tracking capacity of a pre-linguistic child, the verbal expression *five horses* does not (Feigenson, 2011; Miller, 1956). Children therefore could use language together with set manipulations to discover how sets of five can be combined to form larger sets. Second,

<sup>11</sup>The limits on parallel representations of objects vary depending on stimulus and task factors, as well as the development of perceptual skills (e.g., Alvarez & Franconeri, 2007; Green & Bavelier, 2003). So do the limits on ANS acuity, which increases with practice and with learning of mathematics (Piazza et al., 2013). In all cases, however, parallel representations of objects and ANS representations of numerical magnitudes show limits to which the natural number system is not subject.

most languages contain other quantifiers that can capture numerical information, such as *some*, *more*, and *most* (Halberda, Taing, & Lidz, 2008). Learning the meanings of these quantifiers (a challenge in itself) may strengthen children's sense of how distinct and overlapping sets relate to one another.<sup>12</sup>

With these resources, children may take on the task of deciphering the counting routine of their culture, learning its conventional base system, learning Arabic notation and arithmetic algorithms, and memorizing a body of arithmetic facts. Before undertaking these tasks, however, children will have mastered much of the basic logic of the natural number system,<sup>13</sup> encompassing infinitely many exact numbers that obey the laws of addition and multiplication. In particular, children who take this last step are in a position to appreciate that new, larger numbers can be composed from smaller ones through repeated multiplication and addition, including repeated addition of one.<sup>14</sup>

The account that I have sketched preserves the central insights of the two developmental theories that anchor this discussion. As Gelman and Gallistel proposed, exact number concepts are cross-culturally universal and biologically privileged. Because language learning unfolds in a universal pattern, largely without instruction, natural number concepts should develop universally as well. As Carey proposed, moreover, exact number concepts are inaccessible to young children and are acquired gradually. Nevertheless, this account does not root numerical development in the child's engagement with the variable products of its culture, such as a verbal counting procedure, but in culturally universal, innate representations that children begin to combine productively as they master the words and rules of their language. All the information in the system of natural number comes from core systems—the ANS and the systems that together give rise to representations of object kinds—and from the representations and resources that enable learning of nouns and noun phrases, learning quantifiers, and learning rules for combining short phrases to form longer ones whose meanings follow only from the meanings of the words they contain and the rules by which those words combine.

This four-step account, I suggest, makes sense of a considerable body of findings from studies of children's number words and numerical reasoning. First, preschool children solve verbal arithmetic problems when presented with simple noun phrases (e.g., "If there are two bricks in this box and I put in two more, how many bricks are in the box altogether?") well before they can solve such problems when presented with bare number words ("What do two and two make?") (Hughes, 1981, 1986). If number concepts either developed from an innate system of counting principles, or were

<sup>12</sup>One challenge in mastering the meanings of these quantifiers is that they specify inexact quantities, in contrast to the words for natural numbers. Children distinguish number words from these inexact quantifiers early in the process of deciphering number word meanings. Before children have mastered the meanings of *three* and *four*, they know that these words contrast in meaning with one another (a set designated by *three birds* should not be called *four birds*) but not with other quantifiers (a set designated as *three birds* also can be called *some birds*: Condry & Spelke, 2008). Children gain this knowledge before they understand that sets that can be labeled "three birds" vs. "four birds" differ in number (Condry & Spelke, 2008).

<sup>13</sup>I am uncommitted as to whether a child who uses natural language rules to compose new numbers thereby comes to embrace the minimal unit principle, according to which the minimal distance between two numbers is one. Noun phrases allow for expressions such as *five and a half* as well as *five and two more*. Although young children tend to misunderstand these expressions (Gelman, 1991) and older children take a long time to develop a working understanding of fractions (Carey & Spelke, 1994), young children may be open to the possibility that there are numbers beyond those that can be generated by the operations of addition and multiplication, applied recursively to the first three numbers, and therefore that there are numbers that cannot be reached by successive addition of one. Such openness might be a virtue, from the standpoint of mathematical discovery, and it surely is achieved by adults who have discovered the rational, real, and complex numbers. Using natural language, together with core knowledge, children may discover the natural numbers, but perhaps not *only* the natural numbers (see also footnote 14).

<sup>14</sup>Following Chomsky (1988), I therefore propose that children come to command the generative system of natural number by mastering the generative rules of their language. Note, however, that children may master these rules while remaining stunningly ignorant of specific facts of arithmetic, just as adults master recursive rules of multiplication without being able to determine, in specific cases, whether any given number is prime. Moreover, children may come to appreciate that natural number concepts can be generated by successive addition of one long before they realize either that repeated addition of one generates *all* the natural numbers or that is the *only* operation that does so (e.g., that successive additions of *two* will never generate the number *twenty-three*). Thus, children may implement the principle of succession without appreciating its role in defining the natural number system. I thank Veronique Izard for illuminating discussion of this point.

constructed by analogy from the structure of the count list, then arithmetic with bare number words likely would be easier rather than harder for young children.

Second, preschool children's mastery of number words is predicted by the size of their lexicon of nouns. Noun mastery is a better predictor of number word mastery than is age, and its predictive power survives controls for age (Negen & Sarnecka, 2012). The present account provides a natural explanation for these effects. As children learn more nouns, they are in a better position to observe that expressions such as *two pets* refer to arrays that can be named, as well, by expressions such as *the dog and the cat* or *Lassie and Felix*, and that can be composed by the operation of addition (*one pet and one more*).

Third, children whose language makes an obligatory distinction between singular and plural (as in English) learn the meaning of *one* earlier than those whose language does not (Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007). Moreover, children whose language makes an obligatory tripartite distinction between singular, dual and plural learn the meaning of *two* earlier, and remain two-knowers longer, than English-learning children do (Almoammer et al., 2013). These numerical distinctions clearly are not necessary for number word learning (Li, Ogura, Barner, Yang, & Carey, 2009), but grammatical distinctions that pervade language, and that influence the form of the head noun in a noun phrase, nevertheless modulate the pace of children's learning of number words.

The present account was both motivated by, and provides an explanation for, the findings of Izard et al.'s (2014) finger puppet studies and their successors. Once children master the meaning of *three* and map their number words onto ANS representations of cardinal values, they know that the numerical magnitude of a set of two or three finger puppets is changed by the addition or subtraction of a single puppet. Nevertheless, most such children fail to appreciate that the numerical magnitude of a set of six finger puppets will change if the same transformations are applied. Grasp of this insight is predicted by the number of number words in children's vocabulary, independent of age (Jara-Ettinger et al., 2016) as well as by the number of nouns that children have mastered (Negen & Sarnecka, 2012). Mastery of a verbal counting procedure correlates with success on the version of the exact equality task presented to the Tsimane, but it is neither necessary nor sufficient for the attainment of this milestone. All these findings cohere with the view that the system of natural number concepts arises as children master noun phrases composed of diverse nouns and quantifiers, including the first three number words.

This account also may explain both why verbal counting is hard for children to understand—so hard that even children who use counting successfully may do so without understanding the logic behind it (Davidson et al., 2012; Jara-Ettinger et al., 2016)—and how children achieve this understanding. The workings and purpose of counting are obscure to children, because number concepts depend on the words and rules of natural language, and counting procedures do not combine words in the ways that natural language does. Counting rules depend on the linear ordering of words, whereas rules of language do not: they depend on hierarchically structured representations, such as the noun phrases on which I have focused (Chomsky, 1975). Counting also contradicts two basic principles of pragmatics: relevance and economy. In particular, successive pointing to objects fails to provide relevant information to the addressee, because one points at individual objects while referring to the counted subset of objects. And successive recitation of every number word that precedes the relevant one fails to provide information efficiently, because all the relevant information resides in the last word. By learning the meanings of first three number words and the productive combinatorial rules for using these words to compose larger numbers, children become able to understand that new, larger exact numbers can be constructed from known, smaller ones. With this understanding, they can reanalyze verbal counting as a procedure that implements the process of repeated addition and orders the words for successive cardinal values. Thus, I suggest that true understanding of counting is a consequence rather than a cause of children's discovery of the natural numbers.

Finally, this account helps to explain two otherwise puzzling phenomena. First, number-word training studies provide compelling evidence that the count noun context in which a number word appears strongly influences young children's learning of the number word. Yi Ting Huang, Jesse Snedeker, and I attempted to teach children, who had mastered *two*, and who could recite the list of number words up to *ten* or more, the meaning of *three* (Huang et al., 2010), by pairing a single target card depicting three dogs with other cards depicting different numbers of dogs; children were asked to point to the card with "three dogs" and received corrective feedback. When tested with new cards displaying dogs of different sizes or breeds, in different arrangements, children successfully found the cards with three dogs: their learning generalized on the basis of number. In contrast, when tested with cards presenting varying numbers of sheep and asked to find the cards with "three sheep," children pointed to cards with three or more sheep at random. Although the children successfully identified the new cards as depicting sheep, and also found new pictures of "three dogs," they did not combine these abilities so as to find cards presenting three sheep. Children's learning was restricted either to the specific object kind on which they were trained (dogs), or to the specific noun used in the original training (*dogs*).

Research by Sara Cordes distinguished between these possibilities (Posid & Cordes, 2015, Experiment 2). Cordes taught children who had mastered the meaning of *two* the meaning of *three* by pairing pictures of three animals with pictures of one to eight animals and indicating the picture with "three." She presented the same animal species within each pair of pictures and varied the species across training trials (e.g., dogs on one trial and pigs on another trial), but she also varied the noun used to describe each picture during training. When the pictures were labeled "three dogs" or "three pigs," children failed to generalize the number word on test trials with new animals, replicating Huang's findings. In contrast, when the same pictures were labeled "three animals," children successfully generalized to new kinds of animals. Successful application of a newly learned number word therefore depended on the noun with which it appeared.

These findings cast doubt on any theory proposing that children learn the first number word meanings by mapping each number word directly and exclusively to representations of numerical magnitudes generated by the ANS (*contra* Gallistel & Gelman, 1992), to enriched, parallel representations of 1–3 distinct individuals (*contra* Carey, 2009) or to both these representations (*contra* Spelke, 2000).<sup>15</sup> Since infancy, ANS representations have served to enumerate sets of objects of all kinds, as well as sequences of actions and sounds; by early childhood, these representations are precise enough to distinguish arrays of one, two, or three individuals. Since infancy, moreover, infants maintain parallel representations of up to three objects (Feigenson, Carey, & Hauser, 2002; Wynn, 1992a), regardless of the kinds of objects presented (Xu & Carey, 1996). Nevertheless, these representations do not suffice for learning the meaning of *three*.

I now think that children's conservatism in generalizing number words across different noun contexts makes sense. When quantified noun phrases are applied to sets of objects, the correct number word in the phrase varies, depending on the noun: faced with the same group of animals, one may refer to *five animals*, *three dogs*, or *two pigs*. Until children have gained facility with multiple number words, they may do well to restrict their generalization of a new number word to new sets named by the same noun.<sup>16</sup>

<sup>15</sup>This early attempt of mine—to address the limitations of theories of numerical development based exclusively on ANS representations or object representations by drawing on both types of representations, together with the combinatorial resources of natural language—contained logical gaps, because the interface of core object and number representations to language was not clear (see Laurence and Margolis (2005) and Rips et al. (2008) for illuminating discussion). The present account fills some of these gaps but others remain: especially questions concerning the innate representations that allow learning and understanding of natural language expressions concluding quantifiers and conjunction. I hope that research will fill these gaps (see, e.g., Gleitman, 2015; Odic, Pietroski, Hunter, Lidz, & Halberda, 2014).

<sup>16</sup>Izard (personal communication) pointed out that some languages and cultures use different number words for different kinds of things. There are vestiges of such usage in English, where it is natural to talk about *a dozen eggs* or *two dozen donuts* but odd, at least, to speak of *a dozen dollars* or *two dozen thousand people*. Language-specific restrictions on the application of number words may arise as a consequence of language learners' conservative generalization of number words.

Finally, this account may help to explain another well-known and no less puzzling phenomenon: despite the impressive mathematical abilities of infants and animals, children have enormous difficulty learning symbolic arithmetic. For many years, I was perplexed by the difficulties that children experience in learning the basic facts of arithmetic. Mastering exact addition requires learning of only 81 facts. Some facts are closely related to counting (e.g.,  $4 + 1$ ) and most facts are related to other facts by commutativity (e.g.,  $3 + 4 = 4 + 3$ ). Moreover, preschool children can perform approximate arithmetic not only on arrays of dots, but on number symbols (Gilmore et al., 2007): their knowledge of approximate sums (e.g., that the sum of 4 and 2 is closer to 5 than to 10) should serve to constrain learning of addition facts still further. Yet elementary school children take years to master symbolic addition and other aspects of basic arithmetic, including properties as basic as the commutativity of addition (see Rips, Bloomfield, & Asmuth, 2008, for discussion). Why is arithmetic hard to learn, when so much of the information on which it depends is accessible to children before formal instruction begins?

Arithmetic is difficult, I suggest, because of two key features of all natural languages. First, natural languages are productively combinatorial and therefore make infinitely many number concepts available, creating a search problem for the arithmetic learner. Second, natural language syntax and semantics do not reveal the properties of the numerical concepts that exact arithmetic requires. Nothing about the rules of language indicates that *five fives* refers to a number that is larger than *four sixes*. No natural language rules indicate that the expression *two fours* picks out the same number as the expression *four twos* (after all, *my mother's brother* is not the same person as *my brother's mother*). Natural languages even may obscure commutativity (for example, *Mary loves Susan and her brother* has two meanings, whereas *Mary loves her brother and Susan* has one). Natural languages offer children an embarrassment of riches: together with systems of core knowledge, they allow children to represent each of the integers in multiple ways. But natural languages do not reveal the relations of one integer to another or the identity conditions on numerical expressions.

Most children gain a workable understanding of arithmetic, I believe, by means of four devices that most contemporary cultures provide. First, the counting routine, once it is understood, allows children to order and compare the numbers specified by different language expressions. Second, a conventional base system makes the ordering relation among numbers more transparent: translating *five fives* and *four sixes* into base 10 makes it easy to determine that *twenty-five* is greater than *twenty-four*. Third, place value codes for Arabic numerals make the ordering relation visually apparent and convey numerical information economically over a large range of values. Fourth, arithmetic algorithms that build on those codes allow children to perform arithmetic on large numbers by memorizing a smaller set of facts and procedures. All these devices vary across cultures, as Carey observed, and most are learned in school. With these devices, children bring order to the large set of natural number concepts that arise when their combinatorial language is harnessed to generate the natural numbers.

The present account has at least one conspicuous weakness: some of its central features are untested. To my knowledge, no experiment has investigated whether children who have mastered the meanings of *two* and *three* but not *five* interpret phrases such as *three dogs and two cats* or *three pairs* as referring to sets with exact cardinal values. No study has asked whether mastery of such expressions predicts children's discovery of the meaning of words for larger numbers. And no training study has attempted to use such expressions to aid children's discovery of the logic of numerical equality and succession, as it applies to numbers beyond those they have learned. These gaps are serious, because even simple phrases with the structure *x and y* pose interpretive problems for children, for ordinary adults who have mastered their language, and for cognitive scientists.<sup>17</sup>

<sup>17</sup>For example, if John likes the red and blue fish, does he like all the fish that are red or blue, or only the fish that are both red and blue? If John and Mary read a book, how many books were read? If John and Mary met and danced, what licenses the inference that John danced but not that he met? See footnote 15.

Nevertheless, two predictions from this account have begun to be tested through studies of adults. First, monolingual, unschooled adults whose culture lacks any counting routine should nevertheless command the logic of natural number, if their language contains words for the first three numbers and rules for expressing conjunction and quantification. Second, adults who live in a numerate society, but who fail to learn a natural language with these devices, should fail to command the logic of natural number. I end by describing one test of each prediction.

### Natural number concepts without counting

In a preliminary study, Izard, Pica, Spelke, and Dehaene (2008) presented a version of her finger puppets task to a group of unschooled, monolingual adults in an Amazonian group, the Mundurukú. The Mundurukú language has words for ONE, TWO, and THREE, but no other exact cardinal values and no native-language counting routine. Mundurukú speakers use the word for HAND to refer to sets of five, but they apply this word to sets of four to ten objects when asked to enumerate sets in Mundurukú (Pica et al., 2004). Like other Amazonian languages, however, the Mundurukú language allows number words to be combined: one can speak of larger sets as “two hands” or “a hand and two on the side”.

Izard gave Mundurukú adults a stringent test of mastery of natural number concepts, implemented with computer-animated events by an assistant who spoke to the participants only in Mundurukú. The Mundurukú were shown two differently colored rows of 9–11 puzzle pieces, lined up to exhibit one-to-one correspondence relations as far as possible (for example, 10 red pieces interlocked with 10 black pieces, and an 11th black piece appeared with no paired red piece). While the black pieces remained visible, the red pieces moved into an opaque container until all were out of view. At this point, either one red piece was added to the container, one red piece was taken away from it, or both changes occurred in succession. Finally, 9 or 10 red pieces moved out of the container and interlocked with black pieces, and the participant was asked to indicate the container’s remaining contents: did it contain any red pieces?

The Mundurukú succeeded at this task, demonstrating not only a sensitivity to one-to-one correspondence but also an understanding that the set of red pieces in the can had an exact cardinal value that increased by one with the addition of one piece and was restored by the inverse operation. Importantly, these abilities were shown not only by Mundurukú adults who could speak and count in Portuguese, but also by those who could not. Thus, the Mundurukú showed mastery of the logic of natural number (Izard et al., 2008). Nevertheless, because some Mundurukú adults have mastered counting in Portuguese, it is possible that the monolingual adults’ successful performance depended on their experience communicating with the bilingual adults.

In conclusion, studies of competent speakers of Mundurukú provide evidence that quantified noun phrases, number words, and productive numerical expressions are widespread across human languages, and that natural number concepts are widespread as well, even in people who lack a conventional counting routine. They also provide suggestive evidence against Carey’s theory that natural number depends on counting, but the evidence is not conclusive. Studies of the Mundurukú also do not distinguish between Gelman and Gallistel’s account and my own, which differ with respect to the role language plays in the emergence of natural number concepts.<sup>18</sup> The best way to

<sup>18</sup>Studies of speakers of a different Amazonian language, Pirahã, have been argued to adjudicate between these two accounts, providing evidence for the innateness of natural number concepts, but their findings also are inconclusive. The Pirahã language has been described as having no words for any numbers, not even a word for ONE (Frank et al., 2008), and no recursive rules (Everett, 2005), but these descriptions have been disputed (see Everett, 2009; Nevins et al., 2009). Moreover, decisive tests of mastery of the logic of natural number have not yet been performed on the Pirahã. When Pirahã adults were presented with a line of objects and are asked to create, from a new collection of objects, a second line that matched it, they placed objects from the new collection opposite those in the first collection by one-to-one correspondence (Frank et al., 2008). The studies of Izard and Jara-Ettinger show, however, that this performance does not imply that the Pirahã represent the numerical equality of the two sets.

distinguish these two accounts may come from studies of adults with minimal exposure to any conventional language. I end with one such study.

### Number concepts in adults with minimal access to a natural language

Susan Goldin-Meadow has studied in depth the language and cognitive capacities of speakers of *homesign*: congenitally deaf adults who communicate by means of an improvised system of gestures (Goldin-Meadow, 2003). Adult speakers of homesign are rare but can be found in remote communities. Accordingly, Elizabet Spaepen and collaborators (Spaepen, Coppola, Spelke, Carey, & Goldin-Meadow, 2011) investigated the numerical concepts of four such homesigners in rural Nicaragua. All were adults who earned money and were quite well integrated into their local numerate society. They recognized Arabic notation as well as currency and could make appropriate change during financial transactions. All used fingers to communicate about the first three numbers exactly and about larger numbers with reasonable (though not perfect) accuracy.

Nevertheless, the individual homesigners had no canonical gestures for any exact numbers. Although they raised two fingers to describe events involving two objects, they raised different fingers on different occasions. In this respect, their gestures for number differed from the conventional gestures of deaf speakers of sign languages, and they might be better interpreted as two symbols for THING than as one symbol for TWO (Coppola, Spaepen, & Goldin-Meadow, 2013). Homesigners also had no counting routine and performed like young children on number word elicitation tasks (Spaepen et al., 2011). They therefore appeared to lack words for small numbers or combinatorial processes for forming numerical expressions referring to larger numbers.

Further research using a memory span task provided evidence for this suggestion (Spaepen, Flaherty, Coppola, Spelke, & Goldin-Meadow, 2013). When homesigners were presented with a sequence of their own gestures for nouns, verbs and adjectives, they reproduced the sequence as accurately as deaf speakers of Nicaraguan Sign Language (NSL) and hearing Spanish speakers (who, like the homesigners, had no formal education) reproduced the corresponding words in their languages (Spaepen et al., 2013). Thus, homesign gestures for objects, attributes and actions appeared to function as words. Homesigners performed markedly differently from the NSL and Spanish speaking groups, however, when asked to reproduce a sequence of gestures for numbers. Members of the latter groups showed the same memory span for number words as for other words, indicating that the gestures or sounds by which they referred to numbers functioned as single units in memory. Moreover, their memory span for sequences composed of words for larger numbers (e.g., 4-5-4-5-4) was equal to their memory span for sequences of words for smaller numbers (2-2-3-2-3), suggesting that each unit symbolized a single numerical value. In contrast, the homesigners' memory span was markedly lower for number gestures than for other gestures, and their memory span declined with increases in the numerical magnitudes that they gestured. Spaepen et al. (2013) suggested that five raised fingers, used by these homesigners to communicate about five dogs, did not constitute one word for FIVE but five words for DOG or ONE (Spaepen et al., 2013).

If knowledge of natural number can develop without language, then these homesigners, who are experienced at using money and have high exposure to Arabic notation, should be able to form and use exact numerical representations on non-verbal tasks. A series of studies by Spaepen et al. (2011) tested this prediction. For example, the homesigners were asked to engage in a back and forth game, in which the experimenter tapped a participant's arm a given number of times, and the participant reproduced the gesture as accurately as possible. When they received one, two, or three taps, all the homesigners reproduced each number exactly: they evidently understood the task were motivated to perform it correctly, and distinguished among these small numbers as do animals and infants. When they received larger numbers of taps, however, the homesigners reproduced each number with only approximate accuracy. Moreover, no homesigner made any attempt to enumerate taps through the use of fingers or other tally devices, despite their spontaneous use of fingers in communicating about number. Strikingly, the homesigners appeared to appreciate that enumeration should yield an exact

answer, for they struggled with the exact large-number tasks, as if they realized that an approximate answer was not good enough. Nevertheless, they appeared not to understand that the gestures they used in their communications about number (raised fingers) and the visual symbols they used in monetary exchanges (Arabic notation) refer to concepts in the system of natural number. No homesigner was ever observed to use these symbols to establish exact equivalence between two sets of objects or actions.

Spaepen's research suggests that a lifetime spent in a numerate society, using money and communicating with others who possess natural number concepts, does not lead to the emergence of the system of natural number in a person who has not learned a natural language. The structure of homesign gestures does not give homesigners access to the natural number system, and neither does their mastery of Arabic symbols. These symbol systems do not yield the insight that a box containing seven individual objects contains a set with the cardinal value SEVEN, in the absence of a productive natural language with number words and expressions.

Further studies suggest, nevertheless, that people do not need a very rich or elaborate language in order to develop and use natural number concepts. Some adults and children with autism, whose mastery of language often is highly impaired, succeed in learning mathematics and even develop superior calculation skills (Cowan & Frith, 2009), although autism is typically associated with some impairment in mathematical as well as verbal abilities (Aagten-Murphey et al, 2015; Chiang & Lin, 2007). Moreover, adults with aphasia, who have impaired abilities to interpret embedded or passive sentences on the basis of their word order, may show normal abilities to interpret mathematical formulae with similar structures (Varley, Klessinger, Romanowski, & Siegal, 2005). Notably, these adults have a preserved vocabulary of number words, and they combine these words productively. These findings therefore are compatible with the present hypothesis.

Deaf individuals with limited exposure to sign languages also may develop natural number concepts. Dan Hyde, Kevin Shapiro, and others in my lab tested the numerical concepts of an adolescent, IC, who became deaf in early childhood and had received minimal education and minimal contact with any conventional sign language. Nevertheless, IC had learned number words in Portuguese Sign Language and was adept at using these words, together with a set of signs or homesign gestures that designated kinds of objects, to produce noun phrases in which the number word appeared in a constant position (e.g., *dogs-3*; *cups-7*) (Hyde et al., 2011). IC aced every exact numerical task that we presented to him, including those involving Arabic notation and place value (e.g., which number is larger: 4973 or 4793?), failing only on problems that children learn by rote in school (e.g., what is 7 times 6?). The research on IC and on people with autism suggests that the system of natural number can be learned in the absence of extensive experience with a conventional language, if children are able to master names for kinds of objects, number words, and the productive language rules by which such words combine. If they then use these concepts to master counting, the base system of their culture, and visual representations using place value, they may excel at numerical reasoning.

## Overview

As Gelman and Gallistel propose, there is something natural about natural number. Children catch on to the logic of succession and exact equality without being taught, and most human cultures build on that logic by providing a conventional base system, a counting procedure, a spatial symbol system using forms and place value to represent numbers, and a set of arithmetic algorithms. As Carey proposes, moreover, there is something hard about natural number. Indeed, natural number concepts seem to develop even more slowly than her theory predicts, since some children and adults come to use counting procedures for enumeration without grasping the principles of equality and succession. The proposal that natural number depends on natural language builds on both these insights.

This proposal is widely thought to have been disconfirmed by experiments providing evidence for natural number concepts in people whose language includes minimal number words and expressions (Butterworth et al., 2008), and in people with marked language impairments (Varley et al., 2005). Further disconfirmation appears to come from experiments on numerate adults, providing evidence that numerical tasks elicit little detectable activity within the brain regions that support language (Amalric & Dehaene, 2016; Fedorenko, Duncan, & Kanwisher, 2012). These studies do not speak, however, to the hypothesis that natural number concepts arise through the mastery of a small number of basic linguistic devices. The devices to which I appeal capture the logic of natural number but provide little information about each number: such information will come to children only when they learn to count and calculate. Thus, language may provide the key that unlocks this system for us, while exerting only a small influence on the mathematical thinking of adults.

The arguments and evidence presented in this chapter, however, fall short of showing that my proposal is correct. Instead, research suggests that the study of the development of natural number concepts is at an exciting juncture. Developmental cognitive scientists have at least three distinct hypotheses concerning the psychological sources of our discovery of the natural numbers. These hypotheses all have evolved over time, in accord with a panoply of findings concerning children's developing language and thought. Each hypothesis, moreover, is testable through further experiments, especially training studies. Cognitive scientists may be poised to arrive at a better understanding of the sources of some of our simplest and most important abstract concepts.

## Acknowledgments

Thanks to Susan Carey, Julian Jara-Ettinger, and Josh Rule for comments on earlier versions of this manuscript, and to the members of the Carey-Spelke Seminar on Abstract Thought for useful discussions. Special thanks to Veronique Izard for invaluable insights that have influenced my thinking about number for more than ten years, and for penetrating comments, questions, and criticisms relating to the present manuscript.

## Funding

Supported by a grant from NSF (DRL 1348140) and by the NSF-STC Center for Brains, Minds and Machines (CCF-1231216).

## References

- Aagten-Murphy, D., Attucci, C., Daniel, N., Klaric, E., Burr, D., & Pellicano, E. (2015). Numerical estimation in children with autism. *Autism Research*, 8(6), 668–681. doi:10.1002/aur.1482
- Almoammer, A., Sullivan, J., Donlan, C., Marušič, F., Žaucer, R., O'Donnell, T., & Barner, D. (2013). Grammatical morphology as a source of early number word meanings. *Proceedings of the National Academy of Sciences*, 110, 18448–18453. doi:10.1073/pnas.1313652110
- Alvarez, G. A., & Franconeri, S. L. (2007). How many objects can you attentively track? Evidence for a resource-limited tracking mechanism. *Journal of Vision*, 7(13), 1–10. doi:10.1167/7.13.14
- Amalric, M., & Dehaene, S. (2016). Origins of the brain networks for advanced mathematics in expert mathematicians. *Proceedings of the National Academy of Sciences*, 113(18), 4909–4917. doi:10.1073/pnas.1603205113
- Barner, D. (2016, March 9). *Natural language and number word learning*. Paper presented at the Lorentz Workshop on Tightening the Articulation between Language and Number, Leiden, The Netherlands.
- Barth, H., Kanwisher, N., & Spelke, E. S. (2003). The construction of large number representations in adults. *Cognition*, 86(3), 201–221. doi:10.1016/S0010-0277(02)00178-6
- Barth, H., La Mont, K., Lipton, J. S., Dehaene, S., Kanwisher, N., & Spelke, E. S. (2006). Non-symbolic arithmetic in adults and young children. *Cognition*, 98(3), 199–222. doi:10.1016/j.cognition.2004.09.011
- Boot, W. R., Simons, D. J., Stothart, C., & Stutts, C. (2013). The pervasive problem with placebos in psychology: Why active control groups are not sufficient to rule out placebo effects. *Perspectives in Psychological Science*, 8, 445–454. doi:10.1177/1745691613491271
- Brannon, E. M. (2002). The development of ordinal numerical knowledge in infancy. *Cognition*, 83(3), 223–240. doi:10.1016/S0010-0277(02)00005-7

- Brannon, E. M., & Merritt, D. J. (2011). Evolutionary foundations of the approximate number system. In S. Dehaene, & E. M. Brannon (Eds.), *Space, Time, and Number in the Brain: Searching for the Foundations of Mathematical Thought*. New York, NY: Academic Press.
- Brannon, E. M., & Terrace, H. S. (1998). Ordering of the numerosities 1 to 9 by monkeys. *Science*, 282(5389), 746–749. doi:10.1126/science.282.5389.746
- Brooks, N., Audet, J., & Barner, D. (2013). Pragmatic inference, not semantic competence, guides 3-year-olds' interpretation of unknown number words. *Developmental Psychology*, 49(6), 1066–1075. doi:10.1037/a0029384
- Butterworth, B. (1999). *What Counts: How Every Brain is Hardwired for Math*. London, UK: Free Press.
- Butterworth, B., Reeve, R., Reynolds, F., & Lloyd, D. (2008). Numerical thought with and without words: Evidence from indigenous Australian children. *Proceedings of the National Academy of Sciences*, 105(35), 13179–13184. doi:10.1073/pnas.0806045105
- Cantlon, J. F., & Brannon, E. M. (2007). Basic math in monkeys and college students. *Plos Biology*, 5(12), e328. doi:10.1371/journal.pbio.0050328
- Cantlon, J. F., Libertus, M. E., Pinel, P., Dehaene, S., Brannon, E. M., & Pelphrey, K. A. (2009). The neural development of an abstract concept of number. *Journal of Cognitive Neuroscience*, 21(11), 2217–2229. doi:10.1162/jocn.2008.21159
- Carey, S. (2009). *The origin of concepts*. New York, NY: Oxford University Press.
- Carey, S., & Spelke, E. S. (1994). Domain-specific knowledge and conceptual change. In L. Hirschfeld, & S. Gelman (Eds.), *Mapping the mind: Domain specificity in cognition and culture* (pp. 169–200). Cambridge, UK: Cambridge University Press.
- Chiang, H. M., & Lin, Y. H. (2007). Mathematical ability of students with Asperger syndrome and high-functioning autism: A review of literature. *Autism*, 11, 547–556. doi:10.1177/1362361307083259
- Chomsky, N. (1975). *Reflections on language*. New York, NY: Pantheon.
- Chomsky, N. (1988). *Language and problems of knowledge*. Cambridge, MA: MIT Press.
- Condry, K. F., & Spelke, E. S. (2008). The development of language and abstract concepts: The case of natural number. *Journal of Experimental Psychology: General*, 137, 22–38. doi:10.1037/0096-3445.137.1.22
- Coppola, M., Spaepen, E., & Goldin-Meadow, S. (2013). Communicating about number without a language model: Innovation and integration of number devices into homesign grammar. *Cognitive Psychology*, 67, 1–25. doi:10.1016/j.cogpsych.2013.05.003
- Coubart, A., Izard, V., Spelke, E. S., Marie, J., & Streri, A. (2014). Dissociation between small and large numerosities in newborn infants. *Developmental Science*, 17, 11–22. doi:10.1111/desc.12108
- Cowan, R., & Frith, C. (2009). Do calendrical savants use calculation to answer date questions? A functional magnetic resonance imaging study. *Philosophical Transactions of the Royal Society of London B*, 364, 1417–1424. doi:10.1098/rstb.2008.0323
- Crowder, R. G., & Greene, R. L. (2000). Serial Learning: Cognition and Behavior. In E. Tulving, & F. Craik (Eds.), *The Oxford Handbook of Memory*. New York, NY: Oxford University Press.
- Davidson, K., Eng, K., & Barner, D. (2012). Does learning to count involve a semantic induction? *Cognition*, 123(1), 162–173. doi:10.1016/j.cognition.2011.12.013
- de Hevia, M. D., Izard, V., Coubart, A., Spelke, E. S., & Streri, A. (2014). Representations of space, time, and number in neonates. *Proceedings of the National Academy of Sciences*, 111, 4809–4813. doi:10.1073/pnas.1323628111
- de Hevia, M. D., & Spelke, E. S. (2010). Number-space mapping in human infants. *Psychological Science*, 21, 653–660. doi:10.1177/0956797610366091
- Dehaene, S. (2011). *The number sense: How the mind creates mathematics* (2nd ed.). New York, NY: Oxford.
- Dehaene-Lambertz, G., & Spelke, E. S. (2015). The infancy of the human brain. *Neuron*, 88(1), 93–109. doi:10.1016/j.neuron.2015.09.026
- Denison, S., & Xu, F. (2010). Twelve- to 14-month-old infants can predict single-event probability with large set sizes. *Developmental Science*, 13(5), 798–803. doi:10.1111/desc.2010.13.issue-5
- Dillon, M. R., Pires, A. C., Hyde, D. C., & Spelke, E. S. (2015). Children's expectations about training the approximate number system. *British Journal of Developmental Psychology*, 33(4), 411–418. doi:10.1111/bjdp.2015.33.issue-4
- Everett, D. L. (2005). Cultural Constraints on Grammar and Cognition in Pirahã: Another Look at the Design Features of Human Language. *Current Anthropology*, 46(4), 621–646. doi:10.1086/431525
- Everett, D. L. (2009). Pirahã culture and grammar: A reply to some critics. *Language, Journal of the Linguistic Society of America*, 85(2), 405–442. doi:10.1353/lan.0.0104
- Fedorenko, E., Duncan, J., & Kanwisher, N. (2012). Language-selective and domain-general regions lie side by side in Broca's area. *Current Biology*, 22, 2059–2062. doi:10.1016/j.cub.2012.09.011
- Feigenson, L. (2011). Objects, sets, and ensembles. In S. Dehaene, & E. M. Brannon (Eds.), *Attention and Performance, Volume XIV*. Oxford, England: Oxford University Press.
- Feigenson, L., Carey, S., & Hauser, M. (2002). The representations underlying infants' choice of more: Object files versus analog magnitudes. *Psychological Science*, 13(2), 150–156. doi:10.1111/1467-9280.00427
- Feigenson, L., & Halberda, J. (2008). Conceptual knowledge increases infants' memory capacity. *Proceedings of the National Academy of Sciences*, 105(29), 9926–9930. doi:10.1073/pnas.0709884105

- Frank, M. C., Everett, D. L., Fedorenko, E., & Gibson, E. (2008). Number as a cognitive technology: Evidence from Pirahã language and cognition. *Cognition*, *108*(3), 819–824. doi:10.1016/j.cognition.2008.04.007
- Fuson, K. (1988). *Children's counting and concepts of number*. New York, NY: Springer-Verlag.
- Gallistel, C. R. (1990). *The organization of learning*. Cambridge, MA: MIT Press.
- Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition*, *44*(1–2), 43–74. doi:10.1016/0010-0277(92)90050-R
- Gelman, R. (1972). Logical capacity of very young children: Number invariance rules. *Child Development*, *43*(1), 75–90. doi:10.2307/1127873
- Gelman, R. (1991). Epigenetic foundations of knowledge structures: Initial and transcendent constructions. In S. Carey, & R. Gelman (Eds.), *The epigenesis of mind: Essays on biology and cognition* (pp. 293–322). Hillsdale, NJ: Erlbaum.
- Gelman, R., & Gallistel, C. (1978). *The Child's Understanding of Number*. Cambridge, MA: Harvard University Press.
- Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2007). Symbolic arithmetic knowledge without instruction. *Nature*, *447*, 589–591. doi:10.1038/nature05850
- Gleitman, L. (2015, November 13). *The linguistic representation of symmetry*. Keynote address: Boston University Conference on Language Development, Boston, MA.
- Goldin-Meadow, S. (2003). *The resilience of language: What gesture creation in deaf children can tell us about how all children learn language*. New York, NY: Psychology Press.
- Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496–499. doi:10.1126/science.1094492
- Green, C. S., & Bavelier, D. (2003). Action video game modifies visual selective attention. *Nature*, *423*, 534–537. doi:10.1038/nature01647
- Gunderson, E. A., Spaepen, E., & Levine, S. (2015). Approximate number word knowledge before the cardinal principle. *Journal of Experimental Child Psychology*, *130*, 35–55. doi:10.1016/j.jecp.2014.09.008
- Halberda, J., Taing, L., & Lidz, J. (2008). The development of 'most' comprehension and its potential dependence on counting ability in preschoolers. *Language Learning and Development*, *4*(2), 99–121. doi:10.1080/15475440801922099
- Hitch, G. J., Fastame, M. C., & Flude, B. (2005). How is the serial order of a verbal sequence coded? Some comparisons between models. *Memory*, *13*(3–4), 247–258. doi:10.1080/09658210344000314
- Huang, Y. T., Spelke, E., & Snedeker, J. (2010). When is four far more than three? Children's generalization of newly acquired number words. *Psychological Science*, *21*(4), 600–606. doi:10.1177/0956797610363552
- Hughes, M. (1981). Can preschool children add and subtract? *Educational Psychology*, *1*(3), 207–219. doi:10.1080/0144341810010301
- Hughes, M. (1986). *Children and number: Difficulties in learning mathematics*. New York, NY: Wiley-Blackwell.
- Hyde, D. C., Khanum, S., & Spelke, E. S. (2014). Brief non-symbolic, approximate number practice enhances subsequent exact symbolic arithmetic in children. *Cognition*, *131*, 92–107. doi:10.1016/j.cognition.2013.12.007
- Hyde, D. C., & Spelke, E. S. (2009). All numbers are not equal: An electrophysiological investigation of small and large number representations. *Journal of Cognitive Neuroscience*, *21*(6), 1039–1053. doi:10.1162/jocn.2009.21090
- Hyde, D. C., & Spelke, E. S. (2011). Neural signatures of number processing in human infants: Evidence for two core systems underlying numerical cognition. *Developmental Science*, *14*(2), 360–371. doi:10.1111/desc.2011.14.issue-2
- Hyde, D. C., Winkler-Rhoades, N., Lee, S. A., Izard, V., Shapiro, K. A., & Spelke, E. S. (2011). Spatial and numerical abilities without a complete natural language. *Neuropsychologia*, *49*(5), 924–936. doi:10.1016/j.neuropsychologia.2010.12.017
- Hyde, D. C., & Wood, J. N. (2011). Spatial attention determines the nature of nonverbal number representation. *Journal of Cognitive Neuroscience*, *23*(9), 2336–2351. doi:10.1162/jocn.2010.21581
- Izard, V., Pica, P., Spelke, E. S., & Dehaene, S. (2008). Exact equality and successor function: Two key concepts on the path towards understanding exact numbers. *Philosophical Psychology*, *21*(4), 491–505. doi:10.1080/09515080802285354
- Izard, V., Sann, C., Spelke, E. S., & Streri, A. (2009). Newborn infants perceive abstract numbers. *PNAS Proceedings of the National Academy of Sciences*, *106*(25), 10382–10385. doi:10.1073/pnas.0812142106
- Izard, V., Streri, A., & Spelke, E. S. (2014). Towards exact numbers: Understanding exact equality. *Cognitive Psychology*, *72*, 27–53. doi:10.1016/j.cogpsych.2014.01.004
- Jara-Ettinger, J., Piantadosi, S. T., Spelke, E. S., Levy, R., & Gibson, E. (2016). Mastery of the logic of natural numbers is not the result of mastery of counting: Evidence from late counters. *Developmental Science*. doi:10.1111/desc.12459
- Khanum, S., Hanif, R., Spelke, E. S., Berteletti, I., & Hyde, D. C. (2016). Effects of nonsymbolic approximate number practice on symbolic numerical abilities in Pakistani children *PLoS ONE*, *11*(10), e0164436.
- Laurence, S., & Margolis, E. (2005). Number and natural language. In P. Carruthers, S. Laurence, & S. Stich (Eds.), *The innate mind: Structure and content*. New York, NY: Oxford University Press.
- Leslie, A. M., Gelman, R., & Gallistel, C. (2008). The generative basis of natural number concepts. *Trends in Cognitive Sciences*, *12*, 213–218. doi:10.1016/j.tics.2008.03.004

- Li, P., Ogura, T., Barner, D., Yang, S.-J., & Carey, S. (2009). Does the conceptual distinction between singular and plural sets depend on language? *Developmental Psychology*, 45(6), 1644–1653. doi:10.1037/a0015553
- Libertus, M. E., & Brannon, E. M. (2010). Stable individual differences in number discrimination in infancy. *Developmental Science*, 13(6), 900–906. doi:10.1111/desc.2010.13.issue-6
- Lipton, J. S., & Spelke, E. S. (2005). Preschool children's mapping of number words to nonsymbolic numerosities. *Child Development*, 76(5), 978–988. doi:10.1111/cdev.2005.76.issue-5
- McCrink, K., & Wynn, K. (2004). Large-number addition and subtraction by 9-month-old infants. *Psychological Science*, 15(11), 776–781. doi:10.1111/j.0956-7976.2004.00755.x
- McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month-old infants. *Psychological Science*, 18(8), 740–745. doi:10.1111/j.1467-9280.2007.01969.x
- Mehler, J., & Bever, T. G. (1967). Cognitive capacity of very young children. *Science*, 158, 141–142. doi:10.1126/science.158.3797.141
- Miller, G. A. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, 63(2), 81–97. doi:10.1037/h0043158
- Negen, J., & Sarnecka, B. (2012). Number-concept acquisition and general vocabulary development. *Child Development*, 83(6), 2019–2027. doi:10.1111/cdev.2012.83.issue-6
- Nevins, A., Pesetsky, D., & Rodrigues, C. (2009). Pirahã exceptionality: A reassessment. *Language, Journal of the Linguistic Society of America*, 85(2), 355–404.
- Nieder, A., & Dehaene, S. (2009). Representation of number in the brain. *Annual Review of Neuroscience*, 32, 185–208. doi:10.1146/annurev.neuro.051508.135550
- Nieder, A., Freedman, D. J., & Miller, E. K. (2002). Representation of the quantity of visual items in the primate prefrontal cortex. *Science*, 297(5587), 1708–1711. doi:10.1126/science.1072493
- Oakes, L. M., Baumgartner, H. A., Barrett, F. S., Messenger, I. M., & Luck, S. J. (2013). Developmental changes in visual short-term memory in infancy: Evidence from eye-tracking. *Frontiers in Psychology*, 4, 697. doi:10.3389/fpsyg.2013.00697
- Odic, D., Le Corre, M., & Halberda, J. (2015). Children's mapping between number words and the approximate number system. *Cognition*, 138, 102–121. doi:10.1016/j.cognition.2015.01.008
- Odic, D., Pietroski, P., Hunter, T., Lidz, J., & Halberda, J. (2014). Young children's understanding of 'More' and Discrimination of number and surface area. *Journal of Experimental Psychology: Learning, Memory and Cognition*, 39, 451–461.
- Park, J., & Brannon, E. M. (2013). Training the approximate number system improves math proficiency. *Psychological Science*, 24(10), 2013–2019. doi:10.1177/0956797613482944
- Piaget, J. (1952). *The child's conception of number*. London, UK: Routledge.
- Piantadosi, S. T., Jara-Ettinger, J., & Gibson, E. (2014). Children's learning of number words in an indigenous farming-foraging group. *Developmental Science*, 17, 553–563. doi:10.1111/desc.2014.17.issue-4
- Piazza, M., Izard, V., Pinel, P., Le Bihan, D., & Dehaene, S. (2004). Tuning curves for approximate numerosity in the human intraparietal sulcus. *Neuron*, 44(3), 547–555. doi:10.1016/j.neuron.2004.10.014
- Piazza, M., Pica, P., Izard, V., Spelke, E., & Dehaene, S. (2013). Education enhances the acuity of the non-verbal approximate number system. *Psychological Science*, 24(6), 1037–1043. doi:10.1177/0956797612464057
- Piazza, M., Pinel, P., Le Bihan, D., & Dehaene, S. (2007). A magnitude code common to numerosities and number symbols in human intraparietal cortex. *Neuron*, 53(2), 293–305. doi:10.1016/j.neuron.2006.11.022
- Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an amazonian indigene group. *Science*, 306(5695), 499–503. doi:10.1126/science.1102085
- Pinhas, M., Donohue, S. H., Woldorff, M., & Brannon, E. M. (2014). Electrophysiological evidence for the involvement of the approximate number system in preschoolers' processing of spoken number words. *Journal of Cognitive Neuroscience*, 26(9), 1891–1904. doi:10.1162/jocn\_a\_00631
- Posid, T., & Cordes, S. (2015). Verbal counting moderates perceptual biases found in children's cardinality judgments. *Journal of Cognition and Development*, 16, 621–637. doi:10.1080/15248372.2014.934372
- Rips, L. J., Bloomfield, A., & Asmuth, J. (2008). From numerical concepts to concepts of number. *Behavioral and Brain Sciences*, 31, 623–687. doi:10.1017/S0140525X08005566
- Sarnecka, B. W. (2016). How number concepts are like the earth (and unlike faces, loitering, or knitting). In D. Barner, & A. S. Baron (Eds.), *Core knowledge and conceptual change*. New York, NY: Oxford.
- Sarnecka, B. W., Kamenskaya, V. G., Yamana, Y., Ogura, T., & Yudovina, Y. B. (2007). From grammatical number to exact numbers: Early meanings of 'one', 'two', and 'three' in English, Russian, and Japanese. *Cognitive Psychology*, 55(2), 136–168. doi:10.1016/j.cogpsych.2006.09.001
- Siegal, M. (1999). Language and thought: The fundamental significance of conversational awareness for cognitive development. *Developmental Science*, 2, 1–14. doi:10.1111/desc.1999.2.issue-1
- Siegler, R. S., & Shrager, J. (1984). *Strategy choices in addition and subtraction: How do children know what to do? Origins of Cognitive Skills* (pp. 229–293). Hillsdale, NJ: Erlbaum.
- Spaepen, E., Coppola, M., Spelke, E. S., Carey, S. E., & Goldin-Meadow, S. (2011). Number without a language model. *Proceedings of the National Academy of Sciences*, 108, 3163–3168. doi:10.1073/pnas.1015975108

- Spaepen, E., Flaherty, M., Coppola, M., Spelke, E. S., & Goldin-Meadow, S. (2013). Generating a lexicon without a language model: Do words for number count? *Journal of Memory & Language*, 69(4), 496–505. doi:10.1016/j.jml.2013.05.004
- Spelke, E. S. (2000). Core knowledge. *American Psychologist*, 55(11), 1233–1243. doi:10.1037/0003-066X.55.11.1233
- Spelke, E. S. (forthcoming). *Origins of knowledge*.
- Starr, A. B., Libertus, M. E., & Brannon, E. M. (2013). Infants show ratio-dependent number discrimination regardless of set size. *Infancy*, 18(6), 927–941. doi:10.1111/infa.2013.18.issue-6
- Tudusciuc, O., & Nieder, A. (2009). Contributions of primate prefrontal and posterior parietal cortices to length and numerosity representation. *Journal of Neurophysiology*, 101(6), 2984–2994. doi:10.1152/jn.90713.2008
- Varley, R. A., Klessinger, N. J. C., Romanowski, C. A. J., & Siegal, M. (2005). Agrammatic but numerate. *Proceedings of the National Academy of Sciences*, 102(9), 3519–3524. doi:10.1073/pnas.0407470102
- Wagner, J. B., & Johnson, S. C. (2011). An association between understanding cardinality and analog magnitude representations in preschoolers. *Cognition*, 119(1), 10–22. doi:10.1016/j.cognition.2010.11.014
- Wood, J. N., & Spelke, E. S. (2005). Infants' enumeration of actions: Numerical discrimination and its signature limits. *Developmental Science*, 8(2), 173–181. doi:10.1111/desc.2005.8.issue-2
- Wynn, K. (1990). Children's understanding of counting. *Cognition*, 36, 155–193. doi:10.1016/0010-0277(90)90003-3
- Wynn, K. (1992a). Addition and subtraction by human infants. *Nature*, 358(6389), 749–750. doi:10.1038/358749a0
- Wynn, K. (1992b). Children's acquisition of the number words and the counting system. *Cognitive Psychology*, 24, 220–251. doi:10.1016/0010-0285(92)90008-P
- Wynn, K. (1998). Psychological foundations of number: Numerical competence in human infants. *Trends in Cognitive Sciences*, 2(8), 296–303. doi:10.1016/S1364-6613(98)01203-0
- Xu, F. (2007). Sortal concepts, object individuation, and language. *Trends in Cognitive Science*, 11, 400–406. doi:10.1016/j.tics.2007.08.002
- Xu, F., & Carey, S. (1996). Infants' metaphysics: The case of numerical identity. *Cognitive Psychology*, 30(2), 111–153. doi:10.1006/cogp.1996.0005
- Xu, F., & Garcia, V. (2008). Intuitive statistics by 8-month-old infants. *Proceedings of the National Academy of Sciences*, 105(13), 5012–5015. doi:10.1073/pnas.0704450105
- Xu, F., & Spelke, E. S. (2000). Large number discrimination in 6-month-old infants. *Cognition*, 74, B1–B11. doi:10.1016/S0010-0277(99)00066-9