Journal of Experimental Child Psychology 142 (2016) 66–82

Contents lists available at ScienceDirect

Journal of Experimental Child Psychology

journal homepage: www.elsevier.com/locate/jecp

Non-symbolic division in childhood

Koleen McCrink [a,*], Elizabeth S. Spelke [b]

[a] Department of Psychology, Barnard College, Columbia University, New York, NY 10027, USA
[b] Department of Psychology, Harvard University, Cambridge, MA 02138, USA

ARTICLE INFO

Article history:
Received 18 February 2015
Revised 25 June 2015

Keywords:
Number
Division
Cognitive development
Operational momentum
Scaling
Approximate number system

ABSTRACT

The approximate number system (ANS) underlies representations of large numbers of objects as well as the additive, subtractive, and multiplicative relationships between them. In this set of studies, 5- and 6-year-old children were shown a series of video-based events that conveyed a transformation of a large number of objects into one-half or one-quarter of the original number. Children were able to estimate correctly the outcomes to these halving and quartering problems, and they based their responses on scaling by number, not on continuous quantities or guessing strategies. Children's performance exhibited the ratio signature of the ANS. Moreover, children performed above chance on relatively early trials, suggesting that this scaling operation is easily conveyed and readily performed. The results support the existence of a flexible and substantially untrained capacity to scale numerical amounts.
© 2015 Elsevier Inc. All rights reserved.

Introduction

Psychologists who study the development of proportional reasoning have long detailed the difficulty that young children have in understanding symbolic ratios, fractions, and division (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Dixon & Moore, 1996; Fischbein, 1990; Kieren, 1988; Mack, 1990; Moore, Dixon, & Haines, 1991; Nunes, Schliemann, & Carraher, 1993; Piaget & Inhelder, 1956, 1975; Post, 1981; Reyna & Brainerd, 1994; Singer, Kohn, & Resnick, 1997). The part–whole relationship, a key component of proportional reasoning, can be particularly difficult to master

* Corresponding author.
E-mail address: kmccrink@barnard.edu (K. McCrink).

http://dx.doi.org/10.1016/j.jecp.2015.09.015
0022-0965/© 2015 Elsevier Inc. All rights reserved.

in numerical contexts; in one classic study, children up to 7 years of age struggled to understand that an array comprising six roses and two daisies has more flowers than roses (Inhelder & Piaget, 1964). In addition, the inverse relationship between the number of partitions of a set of objects and the number of objects in each partition is challenging for young children (Frydman & Bryant, 1988; Sophian, Garyantes, & Chang, 1997; Spinillo & Bryant, 1991). In one study, 5-to 7-year-olds were presented with a number of "pizza bits" to distribute among multiple sharers with the goal of obtaining the largest share of bits for a "pizza monster" (Sophian et al., 1997). Children had difficulty in realizing that the more portions into which the bits were divided, the fewer the bits in each portion. Finally, preschoolers who see a set of objects divided in half, and who are provided with the number of objects in one of the halves, are unable to infer that the other half has the same number of objects (Frydman & Bryant, 1988). All of these findings suggest that young children fail to grasp the logic of exact division of discrete quantities.

In contrast, a further body of work reveals that young children competently reason about proportions and ratios in tasks presenting continuous quantities such as overall area or space (Duffy, Huttenlocher, & Levine, 2005; Goswami, 1989; Jeong, Levine, & Huttenlocher, 2007; Mix, Levine, & Huttenlocher, 1999; Sophian, 2000; Spinillo & Bryant, 1991; but cf. Piaget & Inhelder, 1956). In one such study, Goswami (1989) found that by 6 years of age children who were given multiple examples of a particular proportion of shaded/unshaded area in a shape were able to pick a test item that showed the same proportion. In a similar paradigm, Sophian (2000) presented 4- and 5-year-olds with a sample stimulus that exhibited a particular proportional relationship (e.g., an animal with a relatively small body and large head) and asked them to choose the animal that was "just like" it from a pair of test items, one of which presented the same proportional relationship. Children successfully performed this analogical task at as young as 4 years across a range of stimulus types and configurations.

There is also suggestive evidence for a special sensitivity to halving within the context of analogical proportion reasoning, with the concept of "half as much" being privileged in the mind of a child. Spinillo and Bryant (1991) presented children with a box that exhibited a particular proportion of one color to another (e.g., 3/8 blue and 5/8 white) along with two pictures and asked children to choose which picture matched the correct proportion exhibited by the box. By 6 years of age, children were significantly better at choosing the matching picture when the foil crossed the half boundary (i.e., 3/8 vs. 5/8) than when it did not (i.e., 3/8 vs. 1/8). They also performed better overall when given a standard stimulus that exhibited an exact one-half proportion. The authors proposed that one-half acts as a category boundary to proportional reasoning. These results, together with other studies that show an early facility with simple proportions that are readily relatable to a proportion of one-half (Ball, 1993; Goswami, 1989; Mix et al., 1999; Singer-Freeman & Goswami, 2001), suggest that the process of mentally halving a continuous amount might be special capacity of limited flexibility.

With few exceptions (e.g., McCrink & Wynn, 2007), young children's successful proportional reasoning has been observed only with non-numerical spatial quantities. Indeed, children seem to be actively impaired in proportional reasoning about precise numbers (Boyer, Levine, & Huttenlocher, 2008; Jeong et al., 2007; Singer-Freeman & Goswami, 2001; Spinillo & Bryant, 1991). For example, Boyer and colleagues (2008) provided first- and third-graders with a standard that exemplified a particular proportion (e.g., a beaker one-third full of juice) and instructed them to choose from a pair of test stimuli the picture that was the "right mix" of juice. Children were able to perform the task readily when the choice was over a continuous amount of liquid, but when the standard was notched into countable units of juice and water even older children failed to intuit and apply the correct relationship between the standard and test items. Dissociations such as these have led some theorists to posit that the strategies children use to reason about number, such as counting, interfere with intuitive notions of proportion (Gelman, Cohen, & Hartnett, 1989; Mix et al., 1999).

A parallel body of literature has established that human adults, children, infants, and non-human animals are able to represent large numbers of objects (Cordes, Gelman, Gallistel, & Whalen, 2001; Gallistel, 1990; Starkey & Cooper, 1980; Starkey, Spelke, & Gelman, 1990; van Loesbrook & Smitsman, 1990; Xu & Spelke, 2000), events (Platt & Johnson, 1971; Wood & Spelke, 2005), and sounds (Lipton & Spelke, 2004; Meck & Church, 1983) in an approximate fashion. The approximate number system (ANS) is thought to be one of several untrained core cognitive systems shaped by natural selection (Dehaene, 1997; Gallistel, 1990). The ability to discriminate two amounts represented by the ANS is determined by their ratio rather than their difference; for example, the discriminability of 10 and 20 is comparable to the discriminability of 40 and 80 (Izard & Dehaene, 2008).

In addition to representing the number of one set of items or events, the ANS allows for the productive combination of numerical quantities. Many populations, including human infants, children, and adults as well as adult non-human animals, can add, subtract, and order approximate numerical magnitudes (Brannon, Wusthoff, Gallistel, & Gibbon, 2001; Cantlon & Brannon, 2006; Capaldi & Miller, 1988). Recent evidence suggests that the suite of available mathematical operations in childhood may also include a process that scales large approximate numbers in a multiplicative or divisive fashion (Barth, Baron, Spelke, & Carey, 2009; McCrink & Spelke, 2010). Barth and colleagues (2009) presented evidence for non-symbolic division of number in a halving task. In this experiment, the authors presented 6- and 7-year-old children who had no formal education in multiplication or division with a large array of objects for a brief amount of time, covered them, played a sound, and showed children a subsequent amount that was diminished at the time of the sound by half its original value. After this training session, children were presented with novel amounts, which were then occluded before the sound was played. A comparison array came down to the side of the transformed and occluded set, and children needed to choose which array was more numerous. Children who saw these transformations were successfully able to anticipate the numerical outcomes and compare this amount with another amount, indicating that they could mentally halve the initial amount prior to learning any division algorithms and without the aid of precise counting. That is, children were able to infer the relationship between 2 and 1 (1/2) and use that relationship to transform a novel magnitude (say, 24 objects) by one-half.

Using a similar paradigm, McCrink and Spelke (2010) presented 5- and 6-year-old children with multiplicative transformations requiring them to scale numerical magnitudes by factors of 2, 2.5, and 4. Children were able to calculate all of these scaling factors, extending Barth and colleagues' (2009) finding that children could double as well as halve magnitudes. Furthermore, the noisiness of the represented outcomes of this multiplicative scaling process increased with increases in the scaling factor either because of the increased size of the represented outcome or because of the increased size of the scaling factor itself. To explain these findings, McCrink and Spelke (2010) posited the existence of an untrained numerical, non-symbolic scaling process that gradually transitions via schooling and rote learning into a faster and more exact system based on verbally encoded stored arithmetic facts. This scaling process is conceptually related to the analogical proportional reasoning tasks over spatial dimensions (e.g., Boyer et al., 2008; Sophian, 2000). The extraction of the scaling change from 1 to 4, and the application of this relationship to derive 40 from 10, relies on similar logic to that shown by children who view a standard rectangle of 1 by 4 units and successfully choose a matching rectangle of 10 by 40 units, as in Goswami's (1989) research on proportional reasoning.

Critically, this non-symbolic numerical scaling process is hypothesized to operate over the abstract variable of number per se—that is, representations from the ANS that arise independently of spatial dimensions such as area that tend to co-vary with magnitude in natural settings. The experiments of Barth and colleagues (2009) and McCrink and Spelke (2010) controlled for non-numerical magnitude information so as to distinguish numerical representations from representations of other quantities such as summed area. In their tasks, moreover, the scaling required that amounts be held in children's memory and transformed serially in an order that follows the serial nature of symbolic written calculations; children needed to store the information about each amount individually in order to intuit the exhibited proportion. This paradigm also allows investigators to examine scaling up and scaling down separately, whereas a match-to-sample task in which both the set and subset are continuously present when calculating the proportion necessarily presents both transformations together (e.g., Goswami, 1989; Jeong et al., 2007; Mix et al., 1999; Sophian, 2000; Spinillo & Bryant, 1991; cf. Boyer et al., 2008). When the set and subset are concurrent, processes of scaling up versus down cannot be separated because it is not clear whether children anchor on the smaller or larger number.

Although many theorists posit that representations arising from the ANS are ultimately distinct from representations of spatial continua (Piazza & Izard, 2009; see Cohen Kadosh & Walsh, 2009, for an opposing view), they nevertheless are related. Spatial continua—such as area and line length—and numerical magnitudes activate both separate and overlapping neurons in rhesus monkeys (*Macaca mulatta;* Tudusciuc & Nieder, 2007), and the two variables productively combine with time to calculate important dimensions such as rate and probability (Gallistel, 1990; Rakoczy et al., 2014). Infants are spontaneously attuned to congruent relationships between number and space (de Hevia, Izard, Coubart, Spelke, & Streri, 2014; de Hevia & Spelke, 2010; Lourenco & Longo, 2010). With respect to proportions, both spatial proportions (e.g., the ratio of two line lengths) and numerical proportions (e.g., the ratio of two Arabic numerals) are readily extracted, related, and generalized, and these operations activate similar brain regions (Jacob & Nieder, 2009a, 2009b; for a review, see Jacob, Vallentin, & Nieder, 2012). Given the positive results documenting flexible computations of spatial proportions in young children (Boyer et al., 2008; Sophian, 2000) and the fundamental relationship between space and non-symbolic number (Lourenco, Bonny, Fernandez, & Rao, 2012), we may observe a similar flexibility of scaling in tasks activating the ANS.

The current investigation aimed to distinguish between two distinct accounts of all these findings. First, non-symbolic division may operate on continuous quantities (e.g., Boyer et al., 2008) but not discrete ones. If this is true, the findings of Barth and colleagues (2009) on non-symbolic halving over number representations were due to the prevalence of one-half as a special mental boundary (e.g., Ball, 1993; Spinillo & Bryant, 1991) and/or of splitting as a central cognitive construct (Confrey, 1994). Second, children's division may operate on non-symbolic approximate quantities for both space and number—but not on exact discrete quantities (leading to the disconnect between notched discrete units and continuous amounts found by Jeong et al., 2007, and Boyer et al., 2008). On the first hypothesis, a conceptual and flexible scaling process will not apply to ANS representations; on the second hypothesis, it will.

Using a paradigm similar to McCrink and Spelke's (2010) study of non-symbolic multiplication allows us to gauge whether a single, bidirectional scaling process supports multiplication and division, such that the mind treats scaling up by a particular factor as the cognitive inverse of scaling down by that same factor. By examining this question, we gain insight into whether a single non-symbolic numerical scaling mechanism provides the platform for two distinct operations. In formal mathematics, scaling up and scaling down are complementary; are non-symbolic scaling abilities complementary as well? Both empirical and anecdotal reports from educators have long noted that there is an asymmetry between children's learning of symbolic multiplication and division during schooling (e.g., Campbell, 1997; Fischbein, Deri, Nello, & Marino, 1985; Mulligan, 1992). Multiplication is considered easier and is usually taught first; the process of learning to divide takes years to master and encompasses difficult concepts such as fractions (Burns, 2007; Rich & Schmidt, 1997). Does non-symbolic numerical scaling also show an asymmetry between multiplication and division, or does the asymmetry arise from the particular nature of the symbolic system and the algorithms for manipulating symbols to achieve the correct arithmetic outcome?

To examine these questions, we adapted the paradigm of McCrink and Spelke (2010) to present 5- and 6-year-old children with non-symbolic division problems in which arrays of varying number were scaled down by a factor of 2.0 (Experiment 1) or 4.0 (Experiment 2). If children cannot divide non-symbolic numerical quantities, and the concept of one-half was somehow special or unique in allowing children to succeed in previous work on the topic (e.g., Barth et al., 2009; Spinillo & Bryant, 1991), we should observe successful performance in the Division by 2 condition and failure in the Division by 4 condition. If, however, a non-symbolic scaling process can take as input any initial amount and scaling factor, and can divide over approximate discrete non-symbolic quantities, we should observe success in both conditions, with lower overall performance as the scaling factor increases. We tracked performance from the first transformation onward in order to obtain a measure of the ease with which children can learn about this relationship from a single exemplar; if halving is especially intuitive, one would predict better performance, more quickly, in the task for that particular proportion. In addition, if non-symbolic division is more difficult than the non-symbolic multiplication found by McCrink and Spelke (2010), we would expect that children in the current study would exhibit worse overall performance when dividing. However, if non-symbolic multiplication and division are psychologically complementary, we should find similar levels of competence for doubling/halving and quadrupling/quartering.

Experiment 1: Division by 2

Method

Participants

A sample of 16 5- and 6-year-old children (8 girls and 8 boys; age range = 63 months 0 days to 83 months 18 days, mean age = 72 months) were recruited via a large mailing database in the greater Boston area of the northeastern United States. Participants were divided into older (72–84 months, $n = 8$) and younger (60–72 months, $n = 8$) age groups. None of the participants in the final sample had formal education in multiplication or division.[1] One additional child was excluded from the final dataset due to experimenter error in recording her answers. This study included 5- and 6-year-olds as the population of interest because they are mature enough to complete the lengthy task, do not yet have formal education on division in school but do have a relatively precise number sense on the values selected here (Halberda & Feigenson, 2008), and were the same age as those tested in McCrink and Spelke (2010), allowing us to compare overall competence in these related studies.

Displays and procedure

Scaling factor introduction. The child and experimenter were seated together in a quiet testing room at a large table and watched the video displays on a Macintosh laptop computer. The child first viewed a display consisting of two blue rectangles, which grew and shrank for several seconds before becoming stationary. As the rectangles remained on the screen, an animated wand appeared from off-screen left and waved over them while making a "magical" twinkling sound. After several seconds of waving, the two rectangles merged into one rectangle and the experimenter exclaimed, "Look! It's our magic dividing wand. It made less. There used to be two blue rectangles, and now there is one. It doesn't matter if the rectangles are big or small. The wand takes two rectangles and makes them one." After this video, the child saw a follow-up transformation video that was identical to the previous video, but now the rectangles were occluded during the waving of the wand. The experimenter paused the movie after the wand waved and asked the child how many rectangles there were behind the screen. Once the child was able to answer this question correctly (on the first trial for 12/16 children and on the second or third trial for the rest), the experiment moved on to the training trials.

Training block. In the training videos, children saw an array of blue rectangles on the left side of the screen. During the first training video, the experimenter pointed to this array and said, "Now we have this many rectangles. There's too many to count, so we're going to concentrate and use our imagination. So, it's not a counting game, it's an imagination game, and we just have to think really hard." After 5 s, an occluder came up from off-screen and occluded the initial array. The dividing wand then came out and waved over the occluded array while the experimenter said, "Look! They're getting divided." A comparison array composed of pink rectangles came down on the right side of the screen. These rectangles were of identical size (1 cm^2) and density ($\sim$15 objects per 25 cm^2) across all training trials. The child was asked to choose where he or she thought there were more rectangles. Critically, the initial array remained occluded while the child made the choice. The participant needed to calculate the inferred outcome behind the occluder and use that outcome to guide the decision as to which array (the transformed array or the comparison array) contained more objects. To control for experimenter bias, the experimenter (a) sat slightly behind the child and several feet from the screen, (b) phrased the question neutrally ("Where do you think there are more?", "Which side of the screen do you think has more?"), and (c) looked at the child, and not the screen, until the child provided an answer. After answering left/right or pointing to the right or left side of the screen, the experimenter recorded the child's answer, *then* pressed a button to drop the occluder and reveal the transformed array, and

[1] The standardized course of instruction for many of the schools in this area is to introduce the terms "quarter" and "half" when children are in first or second grade (e.g., slightly older than our sample). That is not to say that the children did not know these terms or concepts, which occur in conversations with preschool children. In ordinary conversations, however, halving and quartering are more likely to be applied to continuous portions (e.g., "You can have half of this big cookie") than to discrete numbers (e.g., "You can have half of these 10 grapes"). Nevertheless, we avoid the terms entirely when describing the events.

provided feedback to the child as to whether his or her response was correct or incorrect. The next training movie was then presented.

To encourage children to base their judgments on number and not on correlated continuous variables, the outcome array that children viewed at the end of each training trial had an identical area to that of the initial array on that trial, and it was created by joining the pairs of rectangles into one larger rectangle. In this way, children received evidence that the aspect of the display that transformed was number (which was now lesser by a factor of 2.0) and not summed area (which remained the same). The area of the outcome/initial array was, on average, an intermediate value relative to the areas of the comparison arrays. For example, an initial display of 24 objects (~13 cm^2) that was transformed into 12 objects (still ~ 13 cm^2) was shown against a comparison array of 6, 8, 18, and 24 objects (~6, 8, 18, and 24 cm^2, respectively). For these training trials, therefore, the area of the outcome array relative to the area of the comparison array was confounded and children could potentially use one or both variables to arrive at the correct answer. (This confounding variable was controlled in the test trials.)

To test whether children recruited approximate number representations, we manipulated what we term a *distance* factor by providing some test trials in which the relationship between the transformed outcome was relatively disparate (e.g., it differed by a factor of 2.0; the value of the comparison array was either the correct outcome /2.0 or *2.0) or relatively close (e.g., it differed by a factor of 1.5; the value of the comparison array was either the correct amount /1.5 or *1.5). This variation yields a design in which we can examine performance when the correct outcome is psychologically close to (distance /1.5 or *1.5) or far from (distance /2.0 or *2.0) the comparison array, with the prediction that the values used in the distance factor 2.0 trials will be more discriminable, and therefore easier, than those used in the distance factor 1.5 trials. Although the ratio of 1.5:1 between the comparison array and the transformed array may seem insufficiently challenging, research on non-symbolic arithmetic operations in adulthood finds increased errors when combining multiple numerical representations in an equation (Cordes, Gallistel, Gelman, & Latham, 2007; Katz & Knops, 2014; McCrink, Dehaene, & Dehaene-Lambertz, 2007). Furthermore, the serial nature of the paradigm has the potential to result in relatively erroneous representations at the decision point because children must hold the initial and transformed arrays in their memory for several seconds longer than a traditional numerical discrimination paradigm (e.g., Halberda & Feigenson, 2008) and working memory is relatively weak in young children (Alloway, Pickering, & Gathercole, 2006). During pilot testing, this ratio proved to be challenging for children, as it has been in previous research presenting children with non-symbolic addition and subtraction tasks (Barth, La Mont, Lipton, & Spelke, 2005; Gilmore, McCarthy, & Spelke, 2007). After 12 training trials were completed, children had a small break in which they stretched and moved around the room. After this break, the experiment moved on to the test trials.

Testing block. The 16 test trials were similar to the training trials. An array was presented, occluded, and transformed by a stroke of the wand, and then a comparison array came down from off-screen. Children made the "more" judgment just as in training. However, there were three differences between training and testing. First, children did not see the final product of the transformation behind the occluder (it remained covered even after children answered) and they were not told whether they answered correctly or incorrectly; instead, the experimenter provided uniformly positive feedback on each trial ("Great job! You're doing so well. Let's do another!"). Second, the comparison arrays, which had previously been controlled for density and item size (e.g., each object was 1 cm^2, with ~15 objects per 25 cm^2), were controlled at testing for area and contour length. Whereas during training the comparison arrays were composed of an array of uniform small squares, the comparison arrays at test were varied and rectangular, and each array at each distance (correct outcome /2.0, /1.5, *1.5, or *2.0) had equal area and contour length. For example, an initial array of 16 varied rectangles would be occluded and transformed and remain hidden and then would be compared with arrays of 4, 6, 12, and 16, all of which had the same area and contour length as each other (in this case, ~31 cm^2 and 102 cm^2 contour length). If children had come to rely on area or contour length during the training trials as the sole cues to numerosity of the comparison array (higher area and contour length equaled higher number), they would perform at chance on the test trials. In addition, the overall area and contour length for the comparison arrays at testing (~30 cm^2 area and 100 cm^2 contour length average across all arrays) were chosen to be equidistant from both the very large and very small outcome arrays that children may have visualized behind the occluder based on their training experience. Even though children never saw the outcome arrays, this control discourages a strategy of comparing the area or contour length of the imagined outcome array with that of the comparison array. Thus, children must attend to numerical values, and not perceptual variables that co-vary with number, to succeed on the task. Third, new numerical values were used in testing so that children's performance could not reflect rote learning of the values in the training trials. Fig. 1 presents a schematic of the procedure and stimuli.

Because we wished to promote the use of number, in order to examine the role of the ANS in particular during these calculations, a necessary part of this design was to disallow exact counting and encourage approximate number representations. This was done because the process under examination here is hypothesized to occur over represented imprecise numerical variables typically thought to have been generated from the ANS, and if children generated precise representations, we would be studying a different process. Thus, for both the training and testing trials, children were verbally encouraged not to count the arrays. The experimenter also stopped children from counting when he/she saw them tagging each object and moving their lips to number the objects. Finally, and most important, the initial array was occluded before accurate counting could take place in most instances (after only a few seconds), and after the first trial children abandoned any counting, answering quickly in order to advance to the next trial. Very few tried to explicitly count beyond the first trial given these constraints, and even these children gave up the strategy 2 or 3 trials into training. The arrays used in this experiment were extremely large, such that it was a hopeless strategy to enumerate precise amounts given the time allowed (see Table 1 for specific values).

Design. There were 12 training trials and 16 test trials. On equal numbers of trials in each category, the comparison array differed from the outcome by a distance factor of /2.0 (i.e., the correct outcome to the presented problem divided by 2), /1.5, *1.5, or *2.0. For the exact values used in this experiment and Experiment 2, see Table 1.

Results

For each participant, we calculated mean performance on trials whose comparison arrays were a particular distance from the correct outcome (/2.0, /1.5, *1.5, or *2.0). Overall performance during training (81%) and testing (82%) was significantly above chance (one-sample *t*-tests: $ts(15) = 9.07$ and 10.83, both $ps < .001$, one-tailed). Given a total of 28 trials and an alpha level of .05, a binomial test level of above-chance responding is a minimum of 19 trials correct (67.8%, one-tailed). We calculated the number of children whose individual performance was above this chance threshold, and 15 of 16 children met this criterion (15 observed successes with a total *N* of 16, binomial sign test with an alpha of .05, $p < .001$, one-tailed). An analysis of variance (ANOVA) with block (training block or testing block) and distance (/2.0, /1.5, *1.5, or *2.0) as within-participant factors and gender (male or female) and age (60–72 months or 72–84 months) as between-participant factors was performed over these scores. There were no main effects of block, $F(1, 12) = 0.04$, $p = .83$), gender, $F(1, 12) = 1.45$, $p = .25$, or age, $F(1, 12) = 0.01$, $p = .91$. There was a main effect of distance, $F(3, 36) = 6.90$, $p = .001$. Follow-up Bonferroni-corrected pairwise comparisons suggest that this effect was due to significantly lower performance in the *1.5 distance trials (68%), which differed from the /2.0 distance trials (92%; $p < .05$) but not the /1.5 and *2.0 trials (87% and 80%, respectively; $ps = .17$ and .54). The other distance trials did not significantly differ from each other; within-participant contrasts reveal both a significant quadratic (i.e., U-shaped) trend, $F(1, 12) = 7.22$, $p = .02$, which captures the generally higher performance for distance 2.0 compared with distance 1.5, and a linear trend, $F(1, 12) = 11.99$, $p < .01$, which reflects the poorer overall performance when participants were required to judge comparison arrays that were greater than the correct outcome (*2.0 and *1.5) versus smaller than that outcome (/2.0 and /1.5). There was a significant interaction between block and age, $F(1, 12) = 5.58$, $p < .05$. The younger children performed slightly better in training (85%) than testing (77%), whereas the older children performed slightly better in testing (87%) than training (77%). With the exception of the distance *1.5 training trials, children performed significantly better than chance on all testing and training trial types (one-sample *t*-tests: all $ps < .05$). Overall, children performed better when given comparison

Fig. 1. Schematic of the division videos presented to the children.

Table 1
Exact operand and comparison array values used in Experiments 1 and 2.

Exp 1: Division by 2	Exp 2: Division by 4
Training	Training
24/2 vs. 6	48/4 vs. 6
24/2 vs. 8	48/4 vs. 8
24/2 vs. 18	48/4 vs. 18
24/2 vs. 24	48/4 vs. 24
44/2 vs. 11	88/4 vs. 11
44/2 vs. 15	88/4 vs. 15
44/2 vs. 33	88/4 vs. 33
44/2 vs. 44	88/4 vs. 44
56/2 vs. 14	112/4 vs. 14
56/2 vs. 19	112/4 vs. 19
56/2 vs. 42	112/4 vs. 42
56/2 vs. 56	112/4 vs. 56
Testing	Testing
16/2 vs. 4	32/4 vs. 4
16/2 vs. 5	32/4 vs. 5
16/2 vs. 12	32/4 vs. 12
16/2 vs. 16	32/4 vs. 16
32/2 vs. 8	64/4 vs. 8
32/2 vs. 11	64/4 vs. 11
32/2 vs. 24	64/4 vs. 24
32/2 vs. 32	64/4 vs. 32
48/2 vs. 12	96/4 vs. 12
48/2 vs. 16	96/4 vs. 16
48/2 vs. 36	96/4 vs. 36
48/2 vs. 48	96/4 vs. 48
64/2 vs. 16	128/4 vs. 16
64/2 vs. 21	128/4 vs. 21
64/2 vs. 48	128/4 vs. 48
64/2 vs. 64	128/4 vs. 64

arrays that differed by a factor of 2.0 (86% correct) than a factor of 1.5 (77%) (paired-samples *t*-test: $t(15) = 2.75$, $p = .008$, one-tailed) due to the prediction of a Weber ratio signature commonly found in studies that evoke ANS representations).

To test whether children learned this scaling relationship over the course of training, we tabulated each participant's percentage of correct responding for the first and last trials at each distance trial type (correct /2.0, /1.5, *1.5, or *2.0) during the training block. Because the trials were randomized across participants, taking just the last 4 and first 4 training trials may have yielded inaccurate comparisons that varied with how often each distance trial appeared at the beginning or end of training. Thus, the term "first" here refers to the first trials shown of each distance type, and the absolute number of trials on which children received feedback ranged from 4 (the minimum to capture all ratio types) to 8, depending on how their trial randomization proceeded. We examined training trials, and not test trials, because we were interested in questions of (a) spontaneity and speed of scaling factor abstraction, which occurs for the first time in training, and (b) use of absolute and not relative amount to guide responses, information that is only at the initial stages of calculation in training. Performance was above chance for both the first set and last set of trials of each type (73% and 81%, respectively) (one-sample *t*-tests against chance yield $ts(15) = 4.39$ and 5.83, both $ps < .005$, one-tailed), indicating that children did not require much training to understand and apply the halving transformation. There was no significant improvement from the first set of trials to the last set of trials (paired-sample *t*-test: $t(15) = 1.43$, $p = .17$).

Discussion

The results from this experiment indicate that children with no formal training in division were able to perform a non-symbolic division process of halving the numerosity of an array. This accomplishment required little or no training; children were able to perform the task successfully on the very first trials. Children's above-chance performance on this task replicates the report of non-symbolic halving of number by Barth and colleagues (2009) with a population that is 1 year younger and confirms that success at calculating the outcome to these problems does not depend on representations of spatial variables such as area, contour length, density, and item size but rather depends on numerical amount per se. Finally, performance was modulated by the signature ratio limit of the ANS, as in past studies (Barth et al., 2005; Lipton & Spelke, 2004; Pica, Lemer, Izard, & Dehaene, 2004). Children performed better on trials in which the computed outcome was more psychologically distant from the comparison array. Nevertheless, the current experiment does not reveal whether children have a general and flexible ability to divide quantities or an ability that is limited to the simplest possible operation for scaling down numbers—that is, halving. The next experiment begins to distinguish between these possibilities by investigating whether children can learn with equal ease to divide quantities by a different factor, namely 4.

Experiment 2: Division by 4

Experiment 2 used the same method as Experiment 1 except as follows. Participants were 16 5-year-old ($n = 8$) and 6-year-old ($n = 8$) children (8 girls and 8 boys; age range = 60 months 11 days to 75 months 28 days, mean age = 74 months) recruited from the same database as in Experiment 1. None of the participants had participated in Experiment 1 or had formal education in multiplication or division. The introductory movie, training block, and testing block all portrayed or tested events in which quantities were divided by 4 instead of 2. During the introductory movie, the participants saw four rectangles join to form one rectangle. As in Experiment 1, the comparison arrays were equated for item size and density during training and for area and contour length during testing. The quotients across the experiments were held constant; for example, the Experiment 1 equation of 24/2 had an Experiment 2 analog of 48/4. (Thus, all initial values were doubled.) For the particular values used in this experiment, see Table 1.

Results

As in Experiment 1, each participant was given an average score for each set of trials whose comparison arrays were a particular distance from the correct outcome (/2.0, /1.5, *1.5, or *2.0). Overall performance during training (76%) and testing (71%) was significantly above chance (one-sample *t*-tests: $ts(15) = 6.52$ and 5.62, both $ps < .005$, one-tailed). We also calculated the number of children whose individual performance was above chance (with an alpha level of .05) using a test proportion of 67.8% (19/28 trials). Of the 16 children, 10 showed above-chance performance by this criterion (10 observed successes with a total *N* of 16, binomial sign test, $p < .01$). (Note that, as in Experiment 1, the test proportion was not 50%, as is often the case for binomial sign tests, but rather was the more stringent 67.8%). An ANOVA with block (training block or testing block) and distance (/2.0, /1.5, *1.5, or *2.0) as within-participant factors and gender (male or female) and age (60–72 months or

72–84 months) as between-participant factors was performed over these scores. There were no main effects of block, $F(1, 12) = 0.77$, $p = .40$, gender, $F(1, 12) = 0.05$, $p = .83$, or age, $F(1, 12) = 0.85$, $p = .37$.

There was a main effect of distance, $F(3, 36) = 7.99$, $p < .001$. Follow-up Bonferroni-corrected pairwise comparisons revealed that this effect was due to significantly lower performance in the *1.5 distance trials (56%), which differed from the /2.0 distance trials (88%; $p < .05$) but not the /1.5 and *2.0 trials (77% and 73%, respectively; ps = .27 and .17). The /2.0 distance trials were also significantly or marginally significantly higher than the /1.5 trials ($p = .05$) and *2.0 trials ($p = .03$). The /1.5 trials differed marginally from the /2.0 trials but were similar to the other distance trials, and the *2.0 trials differed from the /2.0 trials ($p = .03$). As in Experiment 1, within-participant contrasts reveal a significant linear trend, $F(1, 12) = 11.72$, $p < .01$, reflecting overall better performance in the /2.0 and /1.5 distance trials compared with *1.5 and *2.0 distance trials, as well as a significant quadratic (i.e., U-shaped) trend, $F(1, 12) = 15.04$, $p < .01$, reflecting lower performance for /1.5 and *1.5 trials relative to /2.0 and *1.5 trials. With the exception of the distance *1.5 training and distance *1.5 testing trials, children performed significantly better than chance on all testing and training trial types (one-sample t-tests: all ps < .05; see Fig. 2). Overall, children performed better when given comparison arrays that differed by a factor of 2.0 (80% correct) than a factor of 1.5 (66%) (paired-samples t-test: $t(15) = 3.89$, $p = .001$, one-tailed).

To test whether children learned the quartering operation only over the course of training, we tabulated each participant's percentage of correct responses for the first and last trials at each distance trial type (correct /2.0, /1.5, *1.5, or *2.0) during the training block. Performance was above chance for both the first set and last set of trials at each distance (73% and 81%, respectively) (one-sample t-tests against chance yield $ts(15) = 3.53$ and 6.46, both ps < .005, one-tailed). Children did not show significant improvement from the first trials to the last trials ($t(15) = 1.09$, $p = .29$).

Discussion

The findings of Experiment 2 indicate that non-symbolic division is not limited to the special case of halving. Children were able to mentally quarter estimated numbers of objects after minimal exposure to this transformation. Their performance exhibited the Weber ratio signature of the ANS, with performance falling as ratio of the comparison array and correct outcome approached 1. The pattern of data also indicates that participants overestimated the outcomes to these transformations.

Further analyses

A series of analyses were conducted across the data from the two experiments to address two additional questions. First, did children's success depend on alternative strategies that were not division based? Second, did the operations of halving and quartering differ in difficulty for children? This finding would be considered a signature of the approximate nature of the ANS and would mirror the increased error with larger multiplication factors seen in McCrink and Spelke (2010). We consider each question in turn.

Range-based strategies

Before we can conclude that children truly divided numbers by 2 or 4, we must examine whether children used alternative strategies to solve these problems. Barth and colleagues (2009) found that children who performed doubling in a similar task were sensitive to the relative extremity of comparison values, consistent with the use of a range-based strategy of judging that the comparison array was larger when it was especially large and smaller when it was especially small. That is, if children were tabulating the range of comparison arrays they see during training, they could infer, using the feedback given, that comparison arrays that are on the high end of the range are always the correct answer and comparison arrays that are on the low end are always the incorrect answer. This initial feedback could lead children to select comparison arrays with extremely high or low values as more or less numerous, respectively, irrespective of what happened to the transformed initial value. To test whether the children in the current experiments used this range-based strategy, we performed three sets of analyses.

First, children who followed a range-based strategy should perform at chance on the very first training trials because they have not had experience with the range of values to be presented as comparison arrays. Moreover, children's performance should improve over the course of the training session. As seen in the sections above, neither prediction was confirmed because children in both experiments exhibited above-chance performance on the first set of training trials (73% for both experiments), and one-tailed paired-samples *t*-tests indicated that they showed no improvement over the training trials in either individual experiment. Children also showed no significant improvement in an analysis comparing their performance on early versus late training trials across the two experiments together, although there is a marginal trend toward better performance on later training trials (81% correct performance on later trials, $t(31) = -1.77$, $p = .09$, one-tailed).

Second, a range strategy predicts chance performance on trials with comparison array values that lie in the middle of the range. To test this prediction, participants' scores were averaged for their performance on test trial types in the middle of the range (the eight comparison array values that were not especially large or small). Performance at midrange was 78% for Experiment 1 (one-sample $t(15) = 9.00$, $p < .01$, one-tailed) and 66% for Experiment 2 (one-sample $t(15) = 3.18$, $p < .01$, one-tailed). There is some evidence that children used range information given that their performance on midrange values, although above chance, was still significantly lower than end-range values in both experiments (78% vs. 86% for Experiment 1 and 66% vs. 77% for Experiment 2) (one-tailed *t*-tests: $t(15) = 2.84$ and 1.99, both *p*s < .05). However, the above-chance performance observed on the early trials and the midrange trials in each of these experiments reveals that range-based strategies do not account for children's success on the task.

Third, if children are using range information only, there will be two specific problems per experiment in which their performance should be *significantly below chance:* 16/2 versus 12 and 16/2 versus 16 (for /2.0) and 32/4 versus 12 and 32/4 versus 16 (for /4.0). Because these comparison arrays reflect the smallest set of possible values, if children are using only range information they will be misled into avoiding these values when asked for the more numerous set.

Side-specific strategies

An additional potential source of bias in children's responses is the possibility of a preference for one type of array over another when answering which side has more objects (e.g., the transformed array or the visible comparison array). This is a potential alternate reason for finding poorer performance in the distance *1.5 trials compared with the distance /1.5 trials; if children chose the transformed array as more numerous by default, we would find the same pattern of data. For example, when choosing whether the outcome to 16/2 is more numerous than 5 (a /1.5 distance trial) or 12 (a *1.5 distance trial), a strategy of simply picking the transformed array would yield a correct answer for the /1.5 distance trial and an incorrect answer for the *1.5 distance trial. However, this strategy cannot explain the pattern of overestimation we find. First, it cannot account for the percentages correct in the distance *1.5 trials (which are at 68% and 56% across testing and training for Experiments 1 and 2, respectively); if children were drawn only to the transformed array, they would have performed significantly *below* chance instead of at or marginally above chance. Second, if children had a bias to pick the transformed array, it would be present throughout the experiment, and it is not; children did not exhibit the same tendency on the critical test trials of distance /2.0 and distance *2.0 (91% vs. 81% in Experiment 1 and 84% vs. 72% in Experiment 2) (paired-sample *t*-tests: both *p*s > .05). It is possible that they exhibited this bias only when the critical comparison at test was difficult in the distance 1.5 trials. However, that possibility entails that children had some idea of what was behind the screen and had performed the scaling transformation. For example, when faced with a value of 16 behind the occluder that undergoes a transformation, one would only find a trial with a comparison array of 5 or 12 (correct /1.5 or correct *1.5), challenging if one was comparing this array with the correct outcome of 8. Thus, although we cannot conclusively say that this factor exerted no effect on children's performance, it does not account for any of the principal findings of these experiments.

Testing for Weber signature of non-symbolic numerical scaling

A process of numerical scaling over ANS representations would yield increased error, and subsequently poorer performance, as the divisive factor increased in magnitude from 2 to 4. A paired-samples *t*-test on overall performance indicates that children performed significantly better in Experiment 1 (82%) than Experiment 2 (73%) ($t(15) = 1.89$, $p = .039$, one-tailed). This performance profile provides some evidence for a non-symbolic scaling process that operates over representations of approximate numerical magnitudes in one of two ways. First, the scaling factor itself may be represented approximately, with greater noise in the representation of 4 than of 2. Second, to equate the numbers to be compared, the initial amounts used in Experiment 2 were double those used in Experiment 1, and so the greater error in Experiment 2 may stem from the greater error associated with the larger initial quantities (we return to this point in the General Discussion). On either of these accounts, the increased error provides evidence for the involvement of the ANS.

General discussion

The current experiments provide evidence for a type of non-symbolic division—an untrained and flexible process of scaling down numerical values, independent from continuous extent variables, that can be extended beyond the special case of halving an amount to the case of quartering as well. Performance in both the halving and quartering conditions was subject to the Weber ratio modulation signature, suggesting that the children recruited representations generated by the ANS. This scaling process was over the variable of number per se; the process of joining several rectangles into one rectangle to convey the divisive relationship allowed area and spatial dimensions to remain constant, whereas the critical variable of number was scaled down by a factor of 2.0 (Experiment 1) or 4.0 (Experiment 2). This process suggests that children's scaling is not limited to continuous spatial amounts or to halving of a numerical array; children without formal training in division can flexibly deploy multiple scaling factors over discrete approximate numerical amounts. Furthermore, unlike symbolic scaling, in which there is an asymmetry in performance for multiplication and division, children in this study exhibited halving (82%) and quartering (74%) performance that was indistinguishable when children were tasked with doubling (82%) and quadrupling (69%) in McCrink and Spelke (2010), a pattern that supports the existence of a bi-directional non-symbolic scaling process.

Children overestimated the correct outcomes to the scaling transformations, as shown by their relatively low or even chance-level performance when the comparison array differed from the outcome by *1.5 and a linear trend in both experiments for better performance when comparison arrays were smaller than the correct outcome. For example, after viewing a problem such as 32/2, children would find a comparison array of 24 (the correct outcome *1.5) more confusable than a comparison array of 10 (the correct outcome /1.5), suggesting that they had miscalculated the correct outcome of 16 as something closer to 24. This overestimation speaks to a set of findings on *operational momentum* (OM; McCrink et al., 2007), a phenomenon in which outcomes to numerical transformations are systematically biased in the direction (larger/smaller) of the operation being performed (see also Katz & Knops, 2014; Knops, Viarouge, & Dehaene, 2009; Pinhas & Fischer, 2008). For example, participants generate outcomes to addition problems that are too large and outcomes to subtraction problems that are too small; this is likely due to a combination of factors such as incorrect compression and decompression of logarithmic representations of magnitudes (Chen & Verguts, 2012; but cf. Knops, Dehaene, Berteletti, & Zorzi, 2014), spatial biases associated with the operational terms themselves (Pinhas, Shaki, & Fischer, 2014), and recruitment of saccadic networks that shift visuospatial attention along a mental number line (Knops, Thirion, Hubbard, Michel, & Dehaene, 2009; Knops, Zitzmann, & McCrink, 2013).

Recently, Katz and Knops (2014) found that adults performing non-symbolic multiplication and division exhibit relative over- and underestimation of outcomes, respectively, an OM bias not seen in the current study. In fact, the children here exhibited the opposite pattern to that observed in adults, echoing the findings in Knops and colleagues (2013), who found an opposite OM tendency in 8-year-olds whose reversal was tied to the presence of adult-like attention-reorienting capacities.

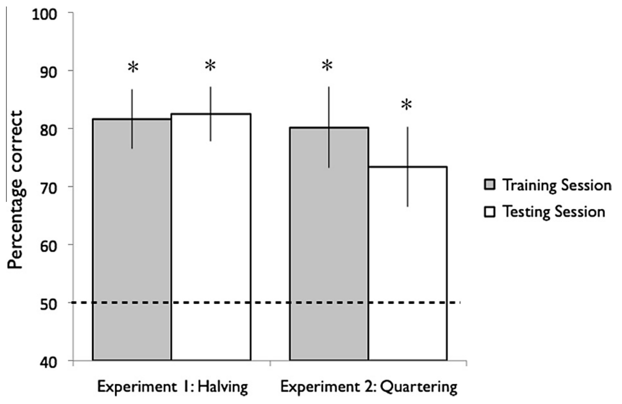

Fig. 2. Performance for the training and testing trials of Experiment 1 (halving task) and Experiment 2 (quartering task). Error bars represent standard errors, and the dotted line indicates chance (50%). An asterisk indicates a < .05 level of significance.

Because we did not have an attention task in the current design, we cannot say whether the same effect drove these findings, but future work on scaling OM in childhood should include an attention-orienting task as a potential variable of analysis to see whether the same pattern as that found by Knops and colleagues holds. If so, this would be evidence that adding, subtracting, multiplying, and dividing draw on a shared representational format such as a mental number line that is bound to children's developing attentional networks.

There is also a clear tendency for children to perform more poorly as the scaling factor moves from one-half to one-quarter. Although it is not possible to pinpoint whether the performance decline is due to the change in initial starting amount, final outcome, or scaling factor, we can examine the patterns in this division study and those reported in McCrink and Spelke (2010) to gain insight as to where the main source of variability is coming from. Between the *2.0 and *4.0 conditions poorer performance could be due to a twofold change in the represented product as well as a twofold change in the scaling factor, and between the /2.0 and /4.0 conditions it could be due to either a twofold change in initial starting amount or a twofold change in scaling factor. However, the *lack* of a performance difference can be telling as well, and here the comparison of performance for children who doubled (in McCrink & Spelke, 2010) and halved (in the current study) becomes germane. The scaling factor (2.0) and outcomes (ranging from 8 to 32) were constant, but the initial amount was four times as large for halving problems. Yet, children's performance was nearly identical (82% for both children in McCrink & Spelke's (2010) doubling condition and children in the current study's halving condition), suggesting that the main source of variability in representation comes from either the outcomes, the scaling factor, or both.

Poorer performance as the scaling factor increases dovetails with work by Boyer and Levine (2012), who presented 6- to 10-year-olds with a proportional analogy task in which children needed to examine a standard (a beak filled with x units juice/y units water) and choose which of two test stimuli exemplified this same proportion. The authors varied how similar in size the test stimuli were to the standard, with some trials exhibiting a low scaling factor (the test and standard were nearly the same size) and some exhibiting a high scaling factor (the test and standard were quite disparate and children needed to scale up or down greatly in order to succeed on the task). They found that children exhibited poorer performance as the scaling factor increased, just as children in the current investigation—and in McCrink and Spelke (2010)—performed more poorly as the scaling factor increased. They also found that there was no difference in scaling up (when the test stimulus was larger than the standard) versus scaling down (when the test stimulus was smaller than the standard).

These similarities raise the possibility that the scaling mechanism found here and in the research of Barth and colleagues (2009) and McCrink and Spelke (2010) is the same as that tapped into during proportional analogy tasks. The process observed here may take as input numerical variables per se—representations generated by the ANS and used to support mathematical reasoning later in life—but it does not do so exclusively. Further research could investigate whether there exists priming or interference between spatial and numerical proportional reasoning, perhaps even despite differences in task specifics, which one would predict if the same mechanism is processing them both, as indicated by research on neural representations of proportions (e.g., Jacob & Nieder, 2009a, 2009b). However, if the partition of labor is such that the ANS both represents and manipulates number only, and a more generic quantity system represents and manipulates spatial dimensions, there might be no such effect.

It may also be the case that the similar pattern of decreased performance with higher scaling factor is due to distinct processes operating in the concrete (e.g., spatial extent) and abstract (e.g., number) domains and having proportional standards that are readily viewable versus remembered and represented. For example, when scaling by a larger factor in Boyer and colleagues' (2008) task, children must overcome greater perceptual dissimilarity when matching a proportional test display to the standard. Thus, in the event of finding no relationship between the tasks, one possible experimental next step would be to adapt the current task to spatial extent only (a la the continuous halving condition in Barth et al., 2009) and determine whether children trained in division over spatial extent are able to generalize this relationship to test items that require division over numerical extent. This experimental design would also help to determine whether one type of scaling (spatial vs. numerical) is more privileged than the other by examining the comparable performance levels during training and ease of generalization at test after being trained on spatial versus numerical arrays.

What does one gain by separating out numerical magnitude per se? It has traditionally been the case that work on the basic science of the ANS, and how children reason about abstract magnitude, has led to discoveries that have impacted the world of applied developmental science. For example, several laboratories have found a link between ANS acuity and certain math abilities later in schooling (Bonny & Lourenco, 2013; Gilmore, McCarthy, & Spelke, 2010; Halberda, Mazzocco, & Feigenson, 2008; Libertus, Feigenson, & Halberda, 2013; Starr, Libertus, & Brannon, 2013). There are now curricula for preschoolers based on early estimation via the ANS (e.g., *Big Math for Little Kids*). Children who are just entering school scaffold up to adding and subtracting using representations computed via the ANS (Gilmore et al., 2007), and children in first grade show faster or more accurate performance of symbolic exact addition after practicing non-symbolic approximate addition of discrete quantities (but not continuous quantities) (Hyde, Khanum, & Spelke, 2014). Given these findings, we believe that it is also important to detail if and how this system supports the fundamental mathematical operations of division and multiplication.

The current evidence for an intuitive understanding of whole-number scaling stands in stark contrast to children's often reported difficulty in learning symbolic division in school (Burns, 2007; Campbell, 1997; Mauro, LeFevre, & Morris, 2003; Rich & Schmidt, 1997; Siegler, 1988). Children were above chance on initial training trials and readily applied the scaling factors of one-half and one-quarter to amounts that extended well beyond those on which they were initially trained. These results suggest that the asymmetry in difficulty between symbolic multiplication and division is due to factors that are specific to the teaching of symbolic division in school. These factors might include the delay in teaching division relative to multiplication, the different ways in which the multiplication and division operations are conceptualized by the teachers and implemented in symbol-manipulating algorithms, or even the emphasis on rote memorization of multiplication but not division.

In fact, the current paradigm represents a step forward in studying proportional relationships at this young age because it requires participants to scale a magnitude in a serial manner and then explicitly generate a representation of the correct outcome. Whereas most proportional reasoning experiments present children with a match-to-sample task, our experiments presented the math problem $x/y = z$, in which participants needed to generate z. This presentation provides a link between the analogical nature of previous work and the serial explicitly generative nature of symbolic problems that children must master when learning division. Given that children use the ANS at the beginning of their symbolic instruction on addition and subtraction (Gilmore et al., 2007), the format presented here could provide a scaffold for reasoning about the relatively difficult concept of fractional understanding in an intuitive fashion based on relations between whole numbers.

These findings, therefore, provide a promising potential avenue for exploration in educational settings and could make the scaling process associated with whole-number dividing into a transparent starting point for discussing what will later be learned symbolically. Because multiplication and division are inverse operations, the animated presentations used in the current experiments might help to make that relationship more transparent. Indeed, because our initial training displays of objects coalescing provided a powerful and concrete image, the use of these displays may provide one reason why this current paradigm was so successful in eliciting good performance on the first training trials. The coalescing of a large number of elements to form a smaller number may be a complement to the "basic conceptual primitive" of splitting proposed by Confrey (1994) as a model of multiplication. Finally, this paradigm dissociates number and amount, which are often directly in conflict in common starting division problems for grade-schoolers (Sophian et al., 1997), allowing children a more flexible grasp of the dimension of number per se as being scaled.

In summary, the current findings support and enrich previous findings of untrained sensitivity to proportional relationships among non-symbolically presented numbers, including previous evidence that children excel at halving (Barth et al., 2009; Goswami, 1989; Mix et al., 1999; Spinillo & Bryant, 1991), that infants represent proportional relationships (Denison & Xu, 2010; Duffy et al., 2005; McCrink & Wynn, 2007; Xu & Denison, 2009), and that non-human animals use rate matching and other proportion-based techniques to optimize foraging (e.g., Harper, 1982; Leon & Gallistel, 1998). They extend the suite of arithmetic abilities that use representations generated by the ANS, illustrate a flexibility with dividing amounts that has not been observed beyond the special case of one-half, and provide a starting conceptual framework that may help children to understand a challenging topic in formal mathematics.

Acknowledgments

This research was supported by National Science Foundation (NSF) Grant DRL-0633955 to the first author and NSF Grants 0633955 and 1348140 to the second author. We are grateful to the children and the parents for their participation and insight during the course of the project.

References

Alloway, T., Pickering, S., & Gathercole, S. (2006). Verbal and visuospatial short-term and working memory in children: Are they separable? *Child Development, 77*, 1698–1716.

Ball, D. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157–195). Hillsdale, NJ: Lawrence Erlbaum.

Barth, H., Baron, A., Spelke, L., & Carey, S. (2009). Children's multiplicative transformations of discrete and continuous quantities. *Journal of Experimental Child Psychology, 103*, 441–454.

Barth, H., La Mont, K., Lipton, J., & Spelke, E. (2005). Abstract number and arithmetic in preschool children. *Proceedings of the National Academy of Sciences, 102*, 14117–14121.

Bonny, J., & Lourenco, S. (2013). The approximate number system and its relation to early math achievement: Evidence from the preschool years. *Journal of Experimental Psychology, 114*, 375–388.

Boyer, T., Levine, S. C., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44*, 1478–1490.

Boyer, T. W., & Levine, S. C. (2012). The development of proportional scaling: Is 1/3 = 2/6 = 3/9 = 4/12? *Journal of Experimental Child Psychology, 111*, 516–533.

Brannon, E., Wusthoff, C., Gallistel, C. R., & Gibbon, J. (2001). Subtraction in the pigeon: Evidence for a linear subjective number scale. *Psychological Science, 12*, 238–243.

Burns, M. (2007). *About teaching mathematics: A K–8 resource* (3rd ed.). Sausalito, CA: Math Solutions.

Campbell, J. (1997). On the relationship between skilled performance of simple division and multiplication. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 23*, 1140–1159.

Cantlon, J., & Brannon, E. (2006). Shared system for ordering small and large numbers in monkeys and humans. *Psychological Science, 17*, 401–406.

Capaldi, E., & Miller, D. (1988). Counting in rats: Its functional significance and the independent cognitive processes that constitute it. *Journal of Experimental Psychology: Animal Behavior Processes, 14*, 3–17.

Carpenter, T., Corbitt, M. K., Kepner, H. S., Jr., Lindquist, M. M., & Reys, R. E. (1981). Decimals: Results and implications from national assessment. *Arithmetic Teacher, 28*(8), 34–37.

Chen, Q., & Verguts, T. (2012). Spatial intuition in elementary arithmetic: A neurocomputational account. *PLoS ONE, 7*(2), e31180.

Cohen Kadosh, R., & Walsh, V. (2009). Numerical representation in the parietal lobes: Abstract or not abstract? *Behavioral and Brain Sciences, 32*, 313–373.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291–330). Albany: State University of New York Press.
Cordes, S., Gallistel, C. R., Gelman, R., & Latham, P. (2007). Nonverbal arithmetic in humans: Light from noise. *Perception & Psychophysics, 69*, 1185–1203.
Cordes, S., Gelman, R., Gallistel, C. R., & Whalen, J. (2001). Variability signatures distinguish verbal from nonverbal counting for both large and small numbers. *Psychonomic Bulletin and Review, 8*, 698–707.
de Hevia, M. D., Izard, V., Coubart, A., Spelke, E., & Streri, A. (2014). Representations of space, time, and number in neonates. *Proceedings of the National Academy of Sciences of the United States of America, 111*, 4809–4813.
de Hevia, M. D., & Spelke, E. S. (2010). Number–space mapping in human infants. *Psychological Science, 21*, 653–660.
Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. New York: Oxford University Press.
Denison, S., & Xu, F. (2010). Twelve- to 14-month-old infants can predict single-event probability with large set sizes. *Developmental Science, 13*, 798–803.
Dixon, J. A., & Moore, C. F. (1996). The developmental role of intuitive principles in choosing mathematical strategies. *Developmental Psychology, 32*, 241–253.
Duffy, S., Huttenlocher, J., & Levine, S. (2005). How infants encode spatial extent. *Infancy, 8*, 81–90.
Fischbein, E. (1990). Intuition and information processing in mathematical activity. *International Journal of Educational Research, 14*, 31–50.
Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3–17.
Frydman, O., & Bryant, P. (1988). Sharing and the understanding of the number equivalence by young children. *Cognitive Development, 3*, 323–329.
Gallistel, C. R. (1990). *The organization of learning*. Cambridge, MA: MIT Press.
Gelman, R., Cohen, M., & Hartnett, P. (1989). To know mathematics is to go beyond the belief that "Fractions are not numbers". In: *Proceedings of Psychology of Mathematics Education* (Vol. 11). North American Chapter of the International Group of Psychology.
Gilmore, C., McCarthy, S., & Spelke, E. (2007). Symbolic arithmetic knowledge without instruction. *Nature, 447*, 589–591.
Gilmore, C., McCarthy, S., & Spelke, E. (2010). Non-symbolic arithmetic abilities and achievement in the first year of formal schooling in mathematics. *Cognition, 115*, 394–406.
Goswami, U. (1989). Relational complexity and the development of analogical reasoning. *Cognitive Development, 4*, 251–268.
Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, 6-year-olds and adults. *Developmental Psychology, 44*, 1457–1465.
Halberda, J., Mazzocco, M., & Feigenson, L. (2008). Individual differences in nonverbal number acuity predict math achievement. *Nature, 455*, 665–668.
Harper, D. (1982). Competitive foraging in mallards: "Ideal free" ducks. *Animal Behaviour, 30*, 575–584.
Hyde, D., Khanum, S., & Spelke, E. (2014). Brief non-symbolic, approximate number practice enhances subsequent exact symbolic arithmetic in children. *Cognition, 131*, 92–107.
Inhelder, B., & Piaget, J. (1964). *The early growth of logic in the child: Classification and seriation* (E. A. Lunzer & D. Papert, Trans.). New York: Harper & Row.
Izard, V., & Dehaene, S. (2008). Calibrating the mental number line. *Cognition, 106*, 1221–1247.
Jacob, S., & Nieder, A. (2009a). Notation-independent representation of fractions in the human parietal cortex. *Journal of Neuroscience, 29*, 4652–4657.
Jacob, S., & Nieder, A. (2009b). Tuning to non-symbolic proportions in the human frontoparietal cortex. *European Journal of Neuroscience, 30*, 1432–1442.
Jacob, S., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: The brain's code for proportions. *Trends in Cognitive Sciences, 16*, 157–166.
Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous vs. discrete quantities. *Journal of Cognition and Development, 8*, 237–256.
Katz, C., & Knops, A. (2014). Operational momentum in multiplication and division? *PLoS ONE, 9*(8), e104777.
Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162–181). Hillsdale, NJ: Lawrence Erlbaum.
Knops, A., Dehaene, S., Berteletti, I., & Zorzi, M. (2014). Can approximate mental calculation account for operational momentum in addition and subtraction? *Quarterly Journal of Experimental Psychology, 67*, 1541–1556.
Knops, A., Thirion, B., Hubbard, E. M., Michel, V., & Dehaene, S. (2009). Recruitment of an area involved in eye movements during mental arithmetic. *Science, 324*, 1583–1585.
Knops, A., Viarouge, A., & Dehaene, S. (2009). Dynamic representations underlying symbolic and nonsymbolic calculation: Evidence from the operational momentum effect. *Attention, Perception, & Psychophysics, 71*, 803–821.
Knops, A., Zitzmann, S., & McCrink, K. (2013). Examining the presence and determinants of operational momentum in childhood. *Frontiers in Psychology, 4*. http://dx.doi.org/10.3389/fpsyg.2013.00325.
Leon, M., & Gallistel, C. R. (1998). Self-stimulating rats combine subjective reward magnitude and subjective reward rate multiplicatively. *Journal of Experimental Psychology: Animal Behavior Processes, 24*, 265–277.
Libertus, M., Feigenson, L., & Halberda, J. (2013). Numerical approximation abilities correlate with and predict informal but not formal mathematics abilities. *Journal of Experimental Child Psychology, 116*, 829–838.
Lipton, J., & Spelke, E. (2004). Discrimination of large and small numerosities by human infants. *Infancy, 5*, 271–290.
Lourenco, S., Bonny, J., Fernandez, E., & Rao, S. (2012). Nonsymbolic number and cumulative area representations contribute shared and unique variance to symbolic math competence. *Proceedings of the National Academy of Sciences of the United States of America, 109*, 18737–18742.
Lourenco, S. F., & Longo, M. R. (2010). General magnitude representation in human infants. *Psychological Science, 21*, 873–881.
Mack, N. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16–32.

Mauro, D. G., LeFevre, J., & Morris, J. (2003). Effects of problem format on division and multiplication performance: Division facts are mediated via multiplication-based representations. *Journal of Experimental Psychology, 29*, 163–170.
McCrink, K., Dehaene, S., & Dehaene-Lambertz, G. (2007). Moving along the number line: Operational momentum in non-symbolic arithmetic. *Perception & Psychophysics, 69*, 1324–1333.
McCrink, K., & Spelke, E. S. (2010). Core multiplication in childhood. *Cognition, 116*, 204–216.
McCrink, K., & Wynn, K. (2007). Ratio abstraction by 6-month-old infants. *Psychological Science, 18*, 740–745.
Meck, W. H., & Church, R. M. (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology: Animal Behavior Processes, 9*, 320–334.
Mix, K., Levine, S. C., & Huttenlocher, J. (1999). Early fraction calculation ability. *Developmental Psychology, 35*, 164–174.
Moore, C. F., Dixon, J. A., & Haines, B. A. (1991). Components of understanding in proportional reasoning: A fuzzy set representation of developmental progressions. *Child Development, 62*, 441–459.
Mulligan, J. (1992). Children's solutions to multiplication and division word problems: A longitudinal study. *Mathematics Education Research Journal, 4*(1), 24–41.
Nunes, T., Schliemann, A. D., & Carraher, D. W. (1993). *Street mathematics and school mathematics*. Cambridge, UK: Cambridge University Press.
Piaget, J., & Inhelder, B. (1956). *The child's conception of space*. New York: Norton.
Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children*. New York: Norton.
Piazza, M., & Izard, V. (2009). How humans count: Numerosity and the parietal cortex. *The Neuroscientist, 15*, 261–273.
Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an Amazonian indigene group. *Science, 306*, 499–503.
Pinhas, M., & Fischer, M. H. (2008). Mental movements without magnitude? A study of spatial biases in symbolic arithmetic. *Cognition, 109*, 408–415.
Pinhas, M., Shaki, S., & Fischer, M. H. (2014). Heed the signs: Operation signs have spatial associations. *Quarterly Journal of Experimental Psychology, 67*, 1527–1540.
Platt, J. R., & Johnson, D. M. (1971). Localization of position within a homogeneous behavior chain: Effects of error contingencies. *Learning and Motivation, 2*, 386–414.
Post, T. R. (1981). Fractions: Results and implications from national assessment. *Arithmetic Teacher, 28*(9), 26–31.
Rakoczy, H., Clüver, A., Saucke, L., Stoffregen, N., Gräbener, A., Migura, J., et al (2014). Apes are intuitive statisticians. *Cognition, 131*, 60–68.
Reyna, V. F., & Brainerd, C. J. (1994). The origins of probability judgment: A review of data and theories. In G. Wright & P. Ayton (Eds.), *Subjective probability* (pp. 239–272). New York: John Wiley.
Rich, B., & Schmidt, P. A. (1997). *Schaum's outline of review of elementary mathematics*. New York: McGraw-Hill.
Siegler, R. S. (1988). Strategy choice procedures and the development of multiplication skill. *Journal of Experimental Psychology, 117*, 258–275.
Singer, J. A., Kohn, A. S., & Resnick, L. B. (1997). Knowing about proportions in different contexts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics* (pp. 115–132). East Sussex, UK: Psychology Press.
Singer-Freeman, K., & Goswami, U. (2001). Does half a pizza equal half a box of chocolates? Proportional matching in an analogy task. *Cognitive Development, 16*, 811–829.
Sophian, C. (2000). Perceptions of proportionality in young children: Matching spatial ratios. *Cognition, 75*, 145–170.
Sophian, C., Garyantes, D., & Chang, C. (1997). When three is less than two: Early developments in children's understanding of fractional quantities. *Developmental Psychology, 33*, 731–744.
Spinillo, A., & Bryant, P. (1991). Children's proportional judgments: The importance of "half". *Child Development, 62*, 427–440.
Starkey, P., & Cooper, R. (1980). Perception of number by human infants. *Science, 210*, 1033–1035.
Starkey, P., Spelke, E. S., & Gelman, R. (1990). Numerical abstraction by human infants. *Cognition, 36*, 97–127.
Starr, A., Libertus, M., & Brannon, E. (2013). Number sense in infancy predicts mathematical abilities in childhood. *Proceedings of the National Academy of Sciences of the United States of America, 110*, 18116–18120.
Tudusciuc, O., & Nieder, A. (2007). Neuronal population coding of continuous and discrete quantity in the primate posterior parietal cortex. *Proceedings of the National Academy of Sciences of the United States of America, 104*, 14513–14518.
van Loesbrook, E., & Smitsman, A. (1990). Visual perception of numerosity in infancy. *Developmental Psychology, 26*, 916–922.
Wood, J. N., & Spelke, E. S. (2005). Infants' enumeration of actions: Numerical discrimination and its signature limits. *Developmental Science, 8*, 173–181.
Xu, F., & Denison, S. (2009). Statistical inference and sensitivity to sampling in 11-month-old infants. *Cognition, 112*, 97–104.
Xu, F., & Spelke, E. (2000). Large number discrimination in 6-month-old infants. *Cognition, 74*, B1–B11.